TEACHING THE UNDERSTANDING OF MATHEMATICS: USING AFFECTIVE CONTEXTS THAT REPRESENT ABSTRACT MATHEMATICAL CONCEPTS

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This paper argues that we do not teach students to understand mathematics. We only teach them mathematics, and leave the understanding - or lack of it - up to them. The reason is that the affective aspects of students' mathematical experiences - feelings that are essential for understanding increasingly abstract mathematical concepts - are, perversely, continually reduced as the mathematics becomes increasingly more abstract. The two major causes of this anomaly are i) the mismatch between the structure of understanding mathematics and its 'logical' structure mirrored in mathematics curricula together with ii) the influence this has, in conjunction with Piagetian theory, on mathematics teaching. The role of inner affective contexts in organising students' mathematical knowledge, developing their mathematical intuition, in their learning and applications of mathematics is discussed. It is also considered how these organisational contexts are conceived and developed, and how to explicitly encouraged this during teaching children how to understand mathematical abstraction.

CONFUSING USES OF 'ABSTRACTION' IN MATHEMATICS AND IN MATHEMATICS EDUCATION

'Abstraction' in mathematics education has been discussed by Dienes (1989), Ginsburg and Asmussen (1988). This is not the same as 'abstraction' as used in mathematics (Nolt 1983). I argue that this currently preferred Piagetian practice in mathematics education is misguided by the confusions of 'abstraction', as used in mathematics and 'abstraction', as used in mathematics education. In particular, mathematics may be described by 'logical' hierarchies e.g. counting comes before addition, which is the foundation for multiplication, which leads on to algebra, which is the basis for ... and so on; or categorisation comes before sets, which are the foundation for groups, which leads on to number systems, which are the bases for ... and so on. These hierarchies are used to describe both mathematics curricula and the steps in understanding - and so teaching - these curricula. Sometimes these steps are described as minute sequential behavioural details, as in the literature on diagnoses and remediation in mathematics education. The variations in possible hierachical descriptions simply illustrates that their 'logic' merely reflects preferences of the practitioners.

Teaching Mathematics From the Cconcrete to the Abstract

'Good' mathematics teaching currently follows the Piagetian stages of development from concrete sensori-motor 'hands-on', through pre-operational and concrete operational stages, to abstract formal operations. Mathematics educationalists traditionally agree that: "In terms of learning mathematics, the ability to cope with abstractions would depend on the emergence or development of formal operational thinking" (Orton, 1992). So 'good' maths teachers introduce concrete objects and use them for the purpose of abstracting mathematics from them. The highest objective of the 'good' maths teacher is to abstract mathematics from real world contexts so that their children understand mathematics as an abstraction in the sense of being context independent. Oh that we could go directly to the abstract rather than being directed by pedagogy through this meandering subterfuge of introducing concrete examples merely so we may later reject them in favour of their abstract mathematical properties. Fortunately the social credibility of teaching through applicable maths makes this 'good' teaching practice more acceptable. For even if our children fail to reach the intended level of abstraction, we have at least served the utilitarianism of human capital education in skilling them for employment.

level, Cartesian coordinates, which depend on an understanding of algebra, are even less understood, and considered even more complex, rarefied and abstract. So abstract describes the difficulty and complexity non-mathematicians associated with increasing hierachical levels of mathematics. However, mathematics educators recognise that mathematics understanding need not follow any such hierachical description of mathematics - for example a child who can find his row and seat number in a theatre, understands much about Cartesian coordinates without necessarily understanding the assumed prerequisites of algebra or even arithmetic. Structural assertions that 'A' must be taught as prerequisite foundation for 'B' are tautological statements generated by ascription to a given 'logical' hierachical description of mathematics. A child centered mathematical educator would rather ask "Does the child now need to know 'B'?" If the answer is yes, then it will be found that the child already has an intuitive understanding of 'A' and the maths teacher may encourage the child to make this explicit. If the child has no intuitive knowledge of 'A' then, from the perspective of a child centered pedagogy, 'A' cannot be taught - only imposed.

Mathematical Abstraction as Context Independent Generalisation

Mathematicians on the other hand, in contrast to non-mathematicians, consider abstraction to be generalisation from particular contexts to context independence. So mathematical abstraction may occur at any level in a descriptive hierarchy. For example, 7 year old's base 10 arithmetic my be generalised to other bases and the properties abstracted for any base 'b'. This meaning of 'abstraction' as generalisation of content and strategies from concrete exemplars to context independence is consistent with Piagetian theory (Dubinsky, 1986) used by our 'good' mathematics teacher and operationalised in mathematics education research (Bettge, 1992; Gonzalez, 1990; Reed, 1989; Iben, 1989; Kouba, 1989 and Cobb, 1987).

Understanding Mathematical Abstraction

I believe that the *understanding* of mathematics does not parallel this 'concrete to abstract' description that is used to describe generalisation in mathematics and used to apply Piagetian theory to the teaching of mathematics - no more than it parallels any particular 'logical' hierachical description of mathematics - as shown above. In particular, as pointed out by Ginsburg and Asmussen (1988), purely cognitive explanations of mathematical thinking ignore the involvement of feelings, personal meanings and motivations in mathematical experiences. I believe that a person's individual understanding of mathematics - no matter how generalised the mathematics - is always organised by an increasingly personalised internal contexts incorporating their mathematical knowledge and beliefs. I suggest that mathematics education should be concerned with how we construct and change these internal contexts.

> In a number of areas there has been a persistent search for units of analysis appropriate for the description of the organization of knowledge and beliefs. In developmental psychology, in problem solving research, in work in artificial intelligence, and in research in expert systems, various approaches have been explored. These can be divided into two main categories: the specific and the general. Among the specific kind are schemata, scripts, and plans. .... Among the general kind are to be found principles, structures, themes, and Keegan's candidate, thought-forms. (Gruber & Davis, 1988, p. 251).

Other approaches are Thematic Appreciation Units - TAUs, Explanation Patterns - XPs (Schank, 1988), Generative Metaphors (Schon, 1979) and Networks of Metaphors (Gruber & Davis, 1988). Another way these approaches may be ordered is on a dimension from non-affective to affect-laden. The traditional view of concrete to abstract parallels affect-laden cognition to affect-free cognition. However, in contradiction to this traditional Piagetian practice in maths education, "highly intellectual individuals engaged in abstract mental activities resort to 'primitive' imagery to solve their problems." (Mavromatis, A. 1987 p. 197). Because these internal organising contexts are so affect-laden I call them 'feeling symbols'. The literature is replete with examples of highly affect-laden organisations for the most abstract concepts in art, literature, science and mathematics.

FEELING SYMBOLS THAT ORGANISE ABSTRACT CONCEPTS IN ART, SCIENCE, LITERATURE AND MATHEMATICS

Feeling symbols in art and literature

Edvard Munch painted The Scream in 1893. The art critic Robert Hughes tells us that Edvard Munch's childhood:

> was ghastly. His father was a ranting religious bigot, his mother a submissive wreck; his beloved sister Sophie died of tuberculosis, and, as he put it later, 'Disease and insanity were the black angels on guard at my cradle. In my childhood I felt always that I was treated in an unjust way, without a mother, sick, and with threatened punishment in Hell hanging over my head.' Thus Munch's main image of family life was the sickroom. (Hughes, 1980, p .277).

Similarly, describing his own feeling symbols, Picasso said: "The painter passes through states of fullness and of emptying. That is the whole secret of art. I take a walk in the forest of Fontainebleau. There I get an indigestion of greenness, I must empty this sensation into a picture. Green dominates in it. The painter paints as if in urgent need to discharge himself of his sensations and his visions" (Ghiselin, 1952, p. 59). In her analysis of Richardson's novel 'Pilgrimage', Wallace (1982) highlights the 'garden' metaphor for women and the 'brow/mouth' metaphor for men that Richardson uses as feeling symbols to illustrate gender differences.

Feeling Symbols in Psychology

"Jung ... often used the images of water to represent the depths of consciousness and of diving and returning (with 'the treasure, the priceless heritage') to represent the growth of self-knowledge and enlightenment." (Shear, 1982, p. 157). Osowski (1986) studied William James' use of metaphor in producing his Principles of Psychology (over the 12 years from 1878 - 1890). Osowski identifies four main metaphors James used as his feeling symbols: stream of thought, flight and perching of a bird, herdsman and fringe of felt relations. Similarly, Gruber and Davis tell us that: "Locke relied on a small set of mutually complementary metaphors for knowledge. Some of the more salient ones were as follows: material object, closed space, acquisition, possession, tool or instrument, and wax tablet." (Gruber & Davis, 1988, p. 259).

Feeling Symbols in Science and Mathematics

Keegan (1985) traces Darwin's thought-form of 'gradualism' in a variety of contexts - change by the accumulation of many small or infinitesimal steps. Keegan suggests that the thought-form of gradualism permeated the whole of Darwin's thinking. Another science example given by Mavromatis (1987) is "Kepler's comparison of the sun, the stars or planets, and the space between them to God the Father, the Son, and the Holy Ghost. .. Kepler's comparison of the planets to the Son is not merely unusual but entirely irrelevant: there is neither external similarity (the Son is one , the planets are many) nor an internal one (the Son does not 'revolve' round the Father)... (yet) ... On one occasion he (Kepler) specifically stressed that 'it is by no means permissible to treat this analogy as an empty comparison; it should be considered by its Platonic form and archetypal quality as one of the primary causes,' " (Mavromatis, 1987, pp .212-213). Gerald Holton (1973) also chronicles many such affect-laden organising themes in science and mathematics around which he has written his book 'Thematic Origins of Scientific Thought: Kepler to Einstein'.

FEELING SYMBOLS - THE INTERNAL CONTEXTS ORGANISING MATHEMATICAL THOUGHT

How feeling symbols organise mathematical thought

It has long been generally recognised in psychiatry, though not in traditional maths education, that affect is central to the organisation of cognition (Ciompi, 1991). We represent mathematical abstractions by these internal affective-cognitive contexts which I call feeling symbols. "Even when dealing with highly abstract concepts, one tends to represent them almost automatically in a way which would render them intuitively accessible." (Fischbein, 1987, p. 212). These feeling symbols influence how we learn mathematics and how we use our mathematical knowledge. "What has been shown in this work is that, beyond the dynamics of the conceptual network, there is a world of stabilized expectations and beliefs which deeply influence the reception and use of mathematical and scientific knowledge." (Fischbein, 1987, p. 206). These feeling symbols direct our mathematical reasoning. "The dynamics of mathematical reasoning - and, generally, of every kind of scientific reasoning - include various psychological components like beliefs and expectations, pictorial prompts, analogies and paradigms. These are not mere residuals of more primitive forms of reasoning. They are genuinely productive, active ingredients of every type of reasoning." (Fischbein, 1987 p. 212). Also: "what is critical to inferential behaviour is the context and goals involved in reasoning, not an abstract logical form it may resemble." (Kuhn, Amsel and O'Loughlin, 1988 p. 19). "Very often, in a reasoning process, the search and solution strategies are influenced by such models functioning tacitly, which are then beyond direct conscious control." (Fischbein, 1987 p. 203).

Why feeling symbols organise mathematical thought

Mathematical feeling symbols condense huge amounts of information and have a substantial identification component. By using empathy and special-kinesthetic representation to understand mathematics, we partially identify with mathematical objects. I think the instinctive aspects of intuitive mathematical thought - speed, lack of conscious effort, personal involvement etc. - result from empathic projection hooking into the drives. (In the case of great intuitions, this is via the cognitive-affective bridge of deep feeling symbols) "condensation mechanisms facilitate drive discharge" (Rothenberg, 1988, p. 70). Our instincts and drives direct our reflexive thoughts and actions which are our mathematical intuitions. Simply, at some level we are the objects of our mathematical understanding. The properties of these objects are our personal characteristics. During this identification, when we take part in a problem we behave according to 'our' characteristics.

How intuitive mathematical understanding develops

Firstly, some people have not learnt to identify with mathematical objects and use spacial-kinesthetic representations of them, so have not developed mathematical intuition, whereas, they may have developed such intuition is other areas. Later I recommend that explicitly teaching such intuitive understanding - via empathic identification and spacial-kinesthetic representations - should be part of mathematics education.

Initial representation of mathematical understanding in the mathematically naive is not the same as the understanding of the trained mathematician (Dubinsky & Lewin, 1986). To understand the initial state and how it changes I will liken the initial naive representation to a small black and white photograph of the exemplars that epitomise the understanding. For example, a child may initially represent the idea of 'two' by remembering the actual two objects that were used to illustrate two. The memory is a close likeness of the objects and there is little involvement in the memory. When many examples of 'two' are known, the life-like photographic representation will change into a more impressionistic picture of, perhaps, the preferred exemplars with the others as shadow images. As more experiences of 'two' cause the memory to be re-constructed with empathic assimilation of the current experiences, the picture further transmogrifies, gradually on each iteration, into a large involving, more encompassing, surrealist colour abstract painting of many exemplars and their many inter-relations - which may not necessarily be mathematical. Using this 'photographic to surrealist" analogy: a deeper understanding of 'two' is like living in an idiosyncratic surrealist world of exemplars, pairs, bonding, mirror symmetries, evenness, parents, Noah's ark, opposites, complementariness, double hops, yin and yang - all one's being is emotionally centered, during the moments of ideation, in the living surrealist world of personal affect-laden connotations of 'two'. If the memory could be re-constructed without assimilation of new experience, then the representation would ossifies rather than transmogrify. Schank's Explanation Patterns - XPs - are similar to such ossified representations: "understanding requires an active memory, full of knowledge based on repeated ossified experiences and also full of novel experiences that are unmerged with other events." (Schank, 1988, p. 221). However, memory never exactly re-constructs itself because there is always a different internal context that co-constructs it. Its more like the game of whispers, where each re-construction slightly changes the previous representation. So repeated memories transform the original experience into a surrealist copy which is the abstract feeling symbol. Albert Rothenberg gives the following example from the long term psychoanalysis of a poet. It shows how, over a long period of creative work - re-constructing the memory and assimilating new relevant experience - the initial feeling symbol can transmogrify.

Similarly, prior to the creation of the metaphor 'the branches were handles of stars', that author had thought only of the sound and shape connections between branches and handles. Afterward, he became dimly aware of images of branchlike maternal arms encompassing a child. During further creative work related to this metaphor, the fire like intensity of the star led to conscious thoughts of warm, erotic sensations and to unearthed unconscious fantasies of erotic sensations in the held child. The unravelling stopped short, however of connecting himself to the held child. (Rothenberg, A. 1988 p. 70)

I believe mature mathematical understandings are, like those above, similarly repeated transmogrifications built on mathematical contexts - e.g. feelingly of motion and perspectives of space - that may have been experienced simultaneously with infantile instincts. Cooney (1991) considers that fundamental intuition of mathematics, the abstraction of the relation of n to n+1, can be understood in terms of such infant experiences. These older mature feeling symbols may contain irrational intentionality contributed by the child's early experiences. Rothenberg (1988) says:

On the basis of recent research on memory and development, there is reason to believe that all childhood events are construed in adulthood in accordance with the child's level of cognitive and affective development at the time they occurred (one citation given). At certain levels of development, for instance, only sensory and motor aspects of an event will be apprehended and experienced. This plays a role in the substance and structure of memories (Rothenberg, 1988, p. 178).

Such mature and pervasive feeling symbols are most widely accessible in regressed primary process thinking. Then they may direct cognitive-affective associations that guide mathematical solutions and strategy selections. Suler (1980) relates that: "Rapaport (1950) in fact described primary process as a drive organization of memory, since all objects, images, and experiences are organized according to their relationship to some instinctual tension." (Suler, 1980, p. 144). "Psychodynamic theory predicts that children who can permit drive-laden material to surface in fantasy and play, and who can cognitively integrate and master that material, should be open to ideas and flexible in their problem-solving approach. Thus, they should be better learners than children who do not actively integrate emotional material." (Russ, 1982, p. 570). This indicates that applications to mathematical education should utilise ego-regressing environments - like storytelling. Only context relevant reduced perspectives of a deep feeling symbol are available in more focused thought.

RESEARCH EXPLORATIONS

To explore the internal contexts subject's use to organise their mathematical knowledge I first asked 80 tertiary maths education students to imagine that they were born again and grew up in 'Flat Land' as either an acute angle or an obtuse angle: Which one they would prefer to be and why? I also asked them to role play their chosen identity with appropriate sounds (not words), body shapes and movements so that the other angles could easily tell which they were. The second part of the investigation was similar except that they had to choose to be one of the following: pi, fractions, division, functions or graphs.

Many subjects represented the open or closed shapes of their chosen angle. Many used cross-model transfer from sharp angle to sharp sounds - and similarly for the sounds made by obtuse angles. Some subjects incorporated the aerodynamic properties of the acute angle in their point first rapid movements, whereas other acute angles moved in rapid, tiny steps. The most interesting representations were complete character identifications - e.g. an acute angle is slim and sophisticated, an obtuse angle is open and friendly.

The second part of the investigation involved the understanding of more abstract mathematical concepts. The results showed, as above, that subjects organise their understanding of the properties of mathematical objects through highly personalised identifications based on empathy and spacial-kinaesthetic representations, sometimes derived from cross-model transfer. In addition, because these mathematical objects were more abstract this investigation allowed for various levels of understanding and different emphases in understanding. The results clearly showed subjects had these differential levels and emphasis of understanding. The lowest levels indicated only symbol recognition - subjects would take the shape of the symbol e.g. pi would stand with legs apart and arms horizontally to the side. The next level of understanding was represented by being an exemplar e.g. a subject folded double would be the fraction 'one-half'. The next higher level of understanding would be one property of the object e.g. the subject who was pi pulling his infinite tail behind him, or an unfinished movement representing a fraction. The highest level of understanding was shown by a holistic representation of a personality whose characteristics were defined the subject's identification with properties of the object e.g. snobbish and haughty, or evil and dangerous. It was interesting that low level symbol representation accompanied movement to represent lack of understanding e.g. a pi that jumps erratically, disappears from here and turns up there. Whereas 'stationary' represented certainty e.g. 'bar-graph' subject was stationary because he was, "more definite, easier to read, easier to get information from me, than if I was moving".

The emphases subjects gave to the representation of their understanding indicated strengths, weaknesses, misconceptions, personal priorities the object held for them and the way their understanding was likely to correctly or incorrectly evolve e.g. a 'fraction' subject shattering does not emphasise the equality of the fractional parts, a 'fraction' subject with independently moving torso and legs emphasised rules to change the top and bottom numbers, a 'fraction' subject with an arm hidden was incomplete.

Another important result was the wealth of metaphors subjects used to organise their understanding at all levels of mathematical abstraction. These could be suggested to other students, when teaching the relevant mathematics, as options for organising new understanding e.g. bar graphs are going up and down stairs.

CONCLUSIONS FOR MATHEMATICS EDUCATION

The construction of mathematics understanding need not follow the 'logical' hierarchy of a curriculum, nor the 'particular to context independent' generalisations of mathematics, nor the 'concrete to formal operational abstraction' of Piagetian pedagogy. In contrast to traditional Piagetian practice it should be recognised that internal affect-laden contexts are essential organisers of intuitive understanding of mathematics at all levels of abstraction, that these feeling symbols drive new learning and mathematical applications. Children should be taught how to develop these personalised contexts through such techniques as encouraging identification with mathematical objects using empathy and spacial-kinaesthetic representations of their properties. At the moment we do not teach children to understand mathematics, we only teach them mathematics and leave the understanding - or lack of it - up to them.

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