TRIADIC SYSTEMS IN EDUCATION: CATEGORICAL, CULTURAL OR COINCIDENCE

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Recent research in education, and mathematics education, in particular, has led to the identification of independent categorizing systems intended to mirror the structures found in such diverse fields as teacher professional development (Barnett, 1992); student writing in mathematics (Clarke, Stephens, & Waywood, in press); and student acquisition of calculus knowledge (Frid, 1992). There are particular characteristics of these categorizing systems which display a tantalising similarity:

- *Contextual similarity - the common location of all three studies within educational environments;*
- *Structural similarity - the "three-valued" (triadic) structure of all three categorizing systems;*
- *Conceptual similarity - categories in each system resemble each other in the nature of their conceptual distinctions.*

This degree of similarity suggests that each categorizing system is an independent manifestation of a more fundamental triadic system (TRIADS). This paper examines the characteristics of these triadic systems and makes comparison with other systems (or analytical frameworks) found in the research or theoretical literature, in an attempt to establish the significance of the degree of conceptual similarity found in the categorizing systems employed in mathematics education. It is proposed that cognitive sophistication be identified with personally contextualized knowledge rather than with formally abstracted knowledge. TRIADS is proposed as a robust structure having relevance in a variety of educational contexts. It is also proposed that conceptual similarities between the first two levels of TRIADS and Skemp's (1976) diadic structure for mathematical understanding support the addition of a third level, to be called Contextual Understanding.

As with other fields, the research endeavour in education is a search for order, pattern and structure. Schoenfeld, among others, has bewailed our lack of "explanatory frames and methods that do *simultaneous* justice to both the cognitive and social aspects of what takes place" in mathematics classrooms (Schoenfeld, 1991, p. 2, emphasis ours). Perhaps it is the perceived need to partition behaviour into affective and cognitive that has made the conceptualization of structure unnecessarily difficult. Beliefs, for instance, lie within the overlap of the cognitive and the affective, and are misrepresented by an analysis which locates them in one domain only. Clarke (1992a) has argued that reductionist approaches to educational research frequently deny the fundamental interrelatedness and socially-situated nature of educational constructs. By way of example, in discussing student adjustment to secondary school mathematics, Clarke (1992b) proposed a single model of "Transition" that could apply to all aspects of behaviour, both cognitive and affective. Certainly, one technique to reveal structure is to study behaviours in diverse contexts, arguing that commonalities across diversity imply fundamental underlying structures. It is the assertion of this paper that such a similarity exists across the behaviour domains of student writing in mathematics, the acquisition of calculus knowledge, and teacher professional development. A case will be made in the following discussion for the generality of the triadic structure proposed for a variety of educational contexts.

Three Empirical Triadic Structures
The archetypal triadic structure (designated "TRIADS") which is being proposed in this paper emerged from comparison of three empirical triadic structures. These three structures, their characteristics, and some details of their empirical origins are outlined in Table 1 and in the following discussion.

Calculus Students: Sources of Conviction
The part of the research study (Frid, 1992) presented here focuses on the nature and role of calculus students' convictions regarding the validity or truth of calculus interpretations and problem responses and the ways students construct their calculus conceptualizations. The term *sources of conviction* is used to refer to how one determines mathematical truth and validity.

The research was a naturalistic study involving three undergraduate calculus classes located at three different post-secondary institutions in Western Canada. Included were a large university and two small private colleges. Task-based personal interviews with 17 students were the method of inquiry into the nature and role of students' *sources of conviction* and manner of construction of calculus conceptualizations. The 12 primary problems given to students asked them to identify, describe, interpret, explain, or apply limit and derivative concepts and they included open-ended as well as relatively focused tasks. The interviews also incorporated relevant personal questions related to students' perceptions of calculus and the learning of calculus, study practices, ways of determining "correctness" and attitudes towards calculus.

Interview data revealed the existence of three groups of students who differed in their *sources of conviction.* These groups were named Collectors, Technicians, and Connectors. The names reflect the nature and role of the groups' *sources of conviction.* Salient characteristics of each of the three groups will now be outlined.

Collectors
Students who from their *sources of conviction* are classified as Collectors display *sources of conviction* that are generally external in nature, in that they reside in statements, rules and procedures presented by the teacher or textbook. They do not generally reside in what students have construed for themselves. The students construct their mathematical knowledge by assembling isolated, relatively unconnected mathematical statements, rules and procedures. Thus, a Collector's calculus conceptualizations can be said to be a "collection" of statements, rules and procedures. Although the student might validly apply calculus knowledge, the student does not claim to know personally whether particular pieces of mathematics are valid or correct. Rather, the student relies on others to determine validity or correctness. These other individuals are perceived by the student to be people for whom calculus is understandable and meaningful.

Technicians
Technician students display a mixture of internal and external *sources of conviction*. Their external *sources of conviction* are similar to Collectors' in that they are based on knowledge of calculus statements, rules and procedures. However, Technicians differ from Collectors in their perception and use of these statements, rules and procedures. Technicians see calculus as a logical organization of statements, rules and procedures and they employ this organization as a technique for thinking about and applying calculus concepts. What therefore most distinguishes Technicians from Collectors is that Technicians display knowledge of how calculus statements, rules and procedures fit together into a logical whole. This logical whole thereby becomes a calculus "technology" in that it is a science or method for thinking about and applying calculus. Technicians can therefore be viewed as skilled users of the application of calculus techniques. Technicians' *sources of conviction* are based upon statements, rules and procedures organized into a coherent, structured set.

Connectors
Connectors generally display *sources of conviction* that are internal in nature. Their *sources of conviction* reside largely in ideas and techniques they perceive to make sense. That is, Connectors view calculus knowledge as something of which they can gain personal understanding and use. They speak of approaching their calculus learning in terms of aiming to understand, make sense of and flexibly think through and apply ideas and techniques. In this way Connectors use their internal *sources of conviction* to construct calculus conceptualizations of which they feel personal understanding. Their conceptualizations are displayed as a network of "connections" between various aspects of calculus and between calculus and themselves.

Student Writing: Stances to Knowledge
This study of student writing was undertaken through the analysis of the mathematics journals of 150 students randomly selected from a school population (Years 7 to 12) of over 500 and through questionnaire completion (Clarke, Waywood and Stephens, in press). During 1988 and 1989, an evaluation was conducted of the consequences of student journal use in school mathematics. The data pertinent to this paper were those concerned predominantly with associated student beliefs and with the nature of student writing related to school mathematics. Consultation with school staff, perusal of a sample of student journals, and the selective interviewing of a cross-section of pupils led to the construction of a questionnaire which, after field testing, was administered to all students in Years 7 to 12. The questionnaire examined student use of journals and their perceptions of the purpose of journal communication and its contribution to their learning of mathematics. Students' conceptions of the nature of mathematics and of mathematical activity in schools were also addressed. While questionnaires were administered to every student, a sample of 150 students, a random sample of 25 at each year level, was chosen for statistical analysis. Three questionnaires were administered ('Mathematics', 'Journals - Part A' and 'Journals - Part B', in that order) and the sample selection procedure ensured that all students at a particular year level, who had completed all three questionnaires, had the same chance of appearing in the sample.

In this study, three *modes of student writing* were identified empirically: Recount, Summary and Dialogue. The categorization of student journal writing, validated by a "blind review" process, was matched with student questionnaire responses regarding the mechanics and the perceived purposes and value of journal use, together with student conceptions of the origin of mathematical ideas, and student perceptions of school mathematics and classroom mathematical activity. Structural consistencies suggested that the triadic structure identified in student writing could also be found in student views regarding the purpose of their journal writing and regarding the nature of mathematical knowledge. The findings of this study suggest that:

- When students write in the *Recount* mode, they see mathematical knowledge as something to be described;
- In the *Summary* mode, students are engaged in integrating mathematical knowledge, now conceived of as a collection of discrete items of knowledge to be collected and connected;
- When writing in the *Dialogue* mode, students are involved in creating and shaping mathematical knowledge, which has now become personalised and purposeful.

The structure of student writing identified by Clarke, Stephens & Waywood (in press) was robust, triadic and hierarchical. The three modes of student writing also appeared to signify three levels of sophistication of mathematical thought. The hierarchical nature of the structure became evident in the analysis of the relationship between mode, student year level, and length of experience with journal completion. A progression from recount to summary to dialogue could be identified, and this progression was associated with the duration of the experience of journal writing rather than simply with year level. The association of each writing mode with a stance towards mathematical knowledge arose from comparison of textual categorizations and questionnaire responses. Those characteristics of the triadic structure pertinent to this paper are shown in Table 1.

Mathematics Teachers: Locus of Authority and Learning Goals
A study by Barnett and Sather (1992) documented the transitions of mathematics teachers who participated in a professional development program based on case discussions. The purpose of the study was to identify changes in teachers' thinking and beliefs about student errors and misunderstandings. Data from pre- and post-interviews were analyzed according to nineteen codes identifying such things as whether a teacher capitalized on student errors as catalysts for discussion or concealed student errors from public scrutiny. Three clusters of codes emerged from the data, and a model was constructed to frame those clusters. The model identifies three levels of progression with regard to teachers' orientations toward authority and their learning goals. The levels appear to be hierarchical and

indicate a general progression from conventional to more reformist points of view. They are briefly characterized below:

- In level 1, the teacher believes that the exclusive authorities in the classroom are the teacher and the text. The learning goal is reception and retention.
- In level 2, students are given some voice, but the final authority resides with the teacher or the text. The learning goal is for students to understand and make connections amongst ideas.
- In level 3, students are the primary authorities. The learning goal is critical analysis and construction of knowledge.

Once the model was constructed, a second analysis was performed to find out if teachers changed levels during the course of the case discussion intervention. In this analysis, each teacher was assigned to a separate level for their pre-intervention and post-intervention interviews. Researchers made their assignments independently by examining their original coding results and by rereading transcripts. The classifications were agreed upon by both researchers for all twenty teachers.

Although the sample is small, the findings suggest that the three levels in the model may be developmental. This is conjectured because each teacher made either no change in level from pre- to post- intervention or advanced from a lower level to a higher level. No teacher moved from a higher level to a lower level. Although the data for this study were focussed on teachers' thinking and beliefs about student errors and misunderstandings, the emergent triadic structure may be indicative of fundamental underlying beliefs that could be identified in other domains of teaching as well.

Table 1. Empirical Triads

Alternative Models			Alternative Conceptual Frameworks			TRIADS Conceptual Similarities			
Sandra Frid - Calculus knowledge	Clarke, Waywood & Stephens - Student Writing	Barnett - Mathematics teaching (professional development)	Source of conviction (Frid)	Stances to knowledge (Clarke/ Waywood/ Stephens)	Locus of authority and learning goals (Barnett)	Perspective	Process	Based in	Located
Collectors	Recount	Explain, simplify	Rules attributable to external authority	Externalized knowledge as facts	Teacher/text is exclusive authority - Goal: reception and retention	External (Predetermined)	Without logic	Unexamined Beliefs	Decontextualized
Technicians	Summary	Discuss, relate	Rules attributable to coherence of logic	Recognition of a system of relationships	Students consulted, teacher/text is final authority. Goal: connection and understanding	Systemic	Logical	Reason	System is the only context
Connectors	Dialogue	Debate, evaluate	Internalized and personalized knowledge	Internalized and personalized knowledge - relativity of truth	Student is final authority. Goal: critical analysis and construction	Personal	Interpretative	Purpose	Contextualized

The first six columns of Table 1 set out the characteristics of the three empirical triadic structures which prompted this paper. The last four columns set out the characteristics of the proposed archetypal TRIADS structure in terms of Perspective, Process, Base, and Location. The next question to be addressed concerns whether TRIADS has relevance in areas beyond the three studies which provide its empirical base.

Other structures

The significance of the emergent similarity in the above structures can be seen in a further comparison with other structures in current and recent educational literature. The contribution and relevance of TRIADS can be delineated by comparison with structures in educational and other contexts.

Piagetian developmental models

The SOLO taxonomy (Biggs & Collis, 1982; Biggs & Collis, 1990) is representative of developmental approaches to modelling cognitive capability, which have built upon the work of Piaget. With other neo-Piagetians, Biggs and Collis usefully distinguish mode of functioning from levels of structural complexity within a mode.

With some reservations, it seems that the neo-Piagetian theorists ... would agree that there are two phenomena involved in determining the level of an individual's response to an environmental cue; the mode of functioning, which is determined by the level of abstraction of the elements utilized, and the complexity of the structure of the response within that mode.
(Collis & Romberg, 1989, pp.7-8)

TRIADS is compatible with the notion of levels as an hierarchical sequence of structural complexity, which is not physiologically based (Collis & Romberg, 1989, p. 13), however the process of progressive refinement of notions of knowledge and associated authority which is modelled in TRIADS places an emphasis on the personalization of knowledge which does not align well with neo-Piagetian models which describe the highest level of sophistication as "Extended Abstract".

Vygotsky's stages of conceptual thought development

Vygotsky's (1962) stages of conceptual thought development form a triadic, hierarchical structure. As children organize their environment by abstracting and labelling perceived qualities of perceived phenomena their conceptual thinking passes through three stages: 1. Arbitrary clustering of experiential phenomena; 2. Organizing experience into complexes using learned modes of organizing and naming phenomena; and, 3. Making personally-derived abstractions.

This interpretation of Vygotsky's third stage melds the personal and the abstract, bringing the categorization scheme in line with TRIADS.

Morality and the development of faith

Kohlberg's stages of moral development (Kohlberg, 1963) are in substantial correspondence with TRIADS. In Kohlberg's picure, Pre-conventional Morality is identified with the existence of an external source of punishment and reward, Conventional Morality invokes societal "laws" as the governing principles, and Post-conventional Morality places priority on an individual's personal sense of "Good", "Just" and "Right". Fowler's theory of faith development (1981) draws upon Kohlberg's work and parallels Kohlberg's stages in the context of the development of religious faith. The correspondence with TRIADS is clear, and can be attributed to similar concerns with the location of authority.

It is possible to find other triadic categorizing structures (for instance, Ernest's (1991) categorization of ideologies into Utilitarian, Purist and Social Change bears some resemblance to the key elements of TRIADS), however the value of TRIADS will ultimately lie with its explanatory and predictive capacity, and it is with a discussion of this that this paper concludes.

Discussion and concluding remarks

Two issues relating to TRIADS warrant discussion:

- the association of the personal with the highest level of sophistication; and,
- the generality of the triadic structure.

Each of these is discussed briefly below.

The Personal or the Abstract

The three studies which have led to the development of TRIADS shared a concern with the way mathematical knowledge was conceptualized. On the surface, any emergent theoretical structures would be located within the cognitive domain. It appears, however, that the correspondence between TRIADS and other theoretical structures is most precise in areas related to *belief*. In fact, the specific conflict which TRIADS establishes with models of cognitive development concerns the tension between the identification of the extremely personal and the extremely abstract as the highest level of sophistication.

Ultimately, theoretical models must be judged according to their utility. Our advocacy of TRIADS as an appropriate and useful model can be supported by reference to previous attempts to understand cognition and, in particular, to understand understanding. The notion of "situated cognition" (Lave, 1988) focussed attention on the contextualized and social nature of learning. Schoenfeld (1992) built upon the notion of cognitive apprenticeship.

> The apprentice tailors are apprenticing themselves into a community, and when they have succeeded in doing so, they have adopted a point of view as well as a set of skills - both of which define them as tailors.
> (Schoenfeld, 1992, p.341)

In the context of mathematics learning, Schoenfeld identified this "point of view" with something called a "mathematical disposition", and this in turn was associated with, among other things, "a predilection to quantify and model" and with "the habit of seeing phenomena in mathematical terms" (Schoenfeld, 1992, p.341). The process which commonly has been associated with this inclination in the past has been that of abstraction of some mathematics from its embeddedness in some problem context. We would suggest rather, that the relevant process should be seen as the learner's previous contextualization of mathematics in an increasing diversity of situations, and that it is this access to multiply-contextualized representations of the relevant mathematics in problematic situations which determines the level of sophistication of the mathematical learning. This personal contextualization of mathematics, which we identify with effective learning, precedes the individual's encounter with a problem solving situation. This view of learning shares some of the features of Sweller's (1989) association of the development of problem solving expertise with effective schema acquisition. It is from this perspective that TRIADS identifies the personal and the contextual with mathematical sophistication.

If the proposed transcendent triadic structure is to be taken seriously and subjected to some sort of test, then one appropriate proving ground would be its application to structures which do not take a triadic form. The descriptive framework first proposed by Skemp (1976) for the categorization of mathematical understanding can be seen to include only the first two of the three categories proposed in this paper. Skemp identified *Instrumental* and *Relational* as categories of understanding and distinguished them as "rules without reasons" and as "knowing both what to do and why". These two categories are entirely consistent with the External and Systemic perspectives proposed in this paper (Skemp, 1992). A consequence of the acceptance of TRIADS would be the identification of a third level of student understanding with the characteristics of the Personal perspective. We propose that Skemp's original framework be extended to include a level with the designation *Contextual*. This level of understanding would be characterized by an ability to articulate mathematical structures in a form that specified a mathematical procedure (for instance) in terms of the context (personal, social, and cultural) in which the mathematics has relevance and meaning. Skemp's formulation of understanding is incomplete in that the Instrumental and Relational forms are appropriately decontextualised, but that the additional dimension of contextually-located personal meaning is omitted entirely.

A student with *Contextual Understanding* knows how to do the mathematics, why it is done that way, and what purpose the mathematics might serve. In practice, the student restricted to instrumental understanding can only mimic taught procedures. The student restricted to relational understanding should be able to solve problems calling for similar mathematical procedures. A student with contextual understanding can employ the mathematical procedure in novel situations in a variety of contexts. It is in this third level of understanding that we find the complex schemas needed for mathematical problem solving. Table 2 sets out TRIADS in relation to Kohlberg's stages of moral development, and the reconceptualized structure for Skemp's (1976) model of understanding.

Table 2. Comparison of TRIADS with Kohlberg (1963) and Skemp (1976)

Alternative Models			Conceptual Frameworks		TRIADS Conceptual Framework			
TRIADS (Clarke, Frid & Barnett, 1993)	Moral Development (Kohlberg, 1963)	Mathematical Understanding (Skemp*, 1976)	Moral Development (Kohlberg, 1963)	Mathematical Understanding (Skemp*, 1976)	Perspective	Process	Based in	Located
EXTERNAL	Pre-Conventional Morality	Instrumental Understanding	Punishment and Reward	Rules without Reasons	External (Predetermined)	Without logic	Unexamined Beliefs	Decontextualized
SYSTEMIC	Conventional Morality	Relational Understanding	Societal "Law and Order"	Knowing both how and why	Systemic	Logical	Reason	System is the only context
PERSONAL	Post-Conventional Morality	*Contextual Understanding**	Personal principles	*Personally contextualized knowledge**	Personal	Interpretative	Purpose	Contextualized

* Skemp's (1976) model of understanding has been augmented to align it with TRIADS.

Categorical, cultural or coincidence

Three alternative explanations can be identified for the degree of correspondence noted above, and each has implications for associated research in education. These alternatives are:

- A single underlying triadic system (TRIADS), of which other triads can be seen to be manifestations within a particular domain of human endeavour, and which should be seen as ultimately informing research in all domains and the understanding of which should constitute the ultimate goal of much educational research (see Eco & Sebeok, 1988);
- A culturally-determined characteristic of the way we categorize our world: three-valued categorizing systems reflecting an inherent inclination for triadic structures which we impose upon all experience - a cautionary reminder that our categories at bottom tell us more about ourselves than they do about the world of our experiences (see Lakoff, 1987);
- A phenomenological coincidence, in which case our understanding of the various categorizing structures is not enhanced by their similarity, and research in the various domains can and should proceed independently of each other (see Clarke, 1992a).

If the first alternative constitutes the best explanation, then an understanding of the characteristics of the underlying "primary triad" (TRIADS) should inform our understanding of the various "secondary triads" both within their individual domains of relevance, and through their common relationship with the primary triad and, consequently, with each other. In this case research should explore the possibility that the study of the behaviour of individuals in one domain could suggest likely characteristics of the behaviour of those individuals in other domains. Such research would have the potential both to inform our understanding of human behaviour and to compel a reconceptualisation of what constitutes an analytic domain. It could emerge that domains such as mathematical behaviour, religious belief and linguistic expression are far more closely interrelated than might have been suspected, and that the study of one can usefully inform the study of the other.

If it could be demonstrated that triadic structures are a consequence of the way people categorize their experiential world, then significant research effort should be expended towards cross-cultural studies of such triadic categorizing systems. The results of such research would inform the interpretation of explanatory frameworks for human behaviour in educational and other settings.

An examination of the many categorizing systems employed in mathematics education, and in education in general, may serve to demonstrate that human behaviour is domain specific, and that an understanding of an individual's mathematical behaviour is neither informed by nor related to an understanding of their behaviour in other domains. In this case, research should be conceived in domain-specific terms and a major goal would be to determine the bounds and characteristics of the hypothesized domains, such as mathematical behaviour.

The research summarised in this paper, and the examples cited above, suggest that triadic structures are in frequent use in educational contexts. Moreover, many of these have strong conceptual similarities. It is our claim that TRIADS is a robust, hierarchical system with the capacity to describe and even explain a wide variety of developmental phenomena in educational, and possibly other, contexts. The implications of the acceptance of TRIADS are profound, particularly with respect to the development of problem solving expertise and the future directions for research into mathematical behaviour.

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