TEACHING MATHEMATICS USING THE PROCEDURAL ANALOGY THEORY

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The role and value of concrete materials in teaching and learning mathematics is uncertain, yet mathematics educators tend to assume their use is essential. Is this an act of faith? This paper describes a Procedural Analogy Theory which attempts to explain the value of concrete materials in the teaching of mathematics. Given the range of teaching possibilities for using concrete materials to help the learning of a particular concept or skill, this theory claims to be able to help teachers develop a teaching approach which will be superior to others. Aspects of both cognitive science and mathematics education are discussed in relation to this theory.

The paper reports on findings when the procedural analogy theory was applied in a number of Year 4 classrooms where Multibased Arithmetic Blocks were used to support the teaching and learning of subtraction algorithms.

An important goal of mathematics education is for students to develop understanding of what they learn in school mathematics, to move beyond the answer to a problem. A pedagogical approach widely assumed to support this goal involves the use of concrete materials in teaching and learning. The use of such materials has been touted in teacher education programs, mathematics education textbooks, school curriculums and academic papers with an intensity of purpose that sometimes approaches missionary zeal. Unfortunately anecdotal accounts, systematic observation of classroom practices, and research in general have been unable to explain the value and role of concrete materials with any degree of certainty. Mathematics educators appear to have set aside this question, perhaps in the belief that it has been answered, or perhaps because there are more fundamental questions that need to be addressed. Lamentably, the answer seems to lie more in human frailty of pursuing interesting ideas, endeavouring to keep current and to find more intriguing questions rather than attempting to find answers to pragmatic problems. Given the prosperous days of mathematics education in the 1960s and 70s, and given the quality of much research in mathematics education since that time, it is disappointing to find ourselves in such a position.

This paper analyses aspects of cognitive science and mathematics education to explain the rationale and the detail of the Procedural Analogy Theory. This theory is intended to answer a range of questions about the effectiveness of concrete materials in teaching and learning. In particular, the theory is a theory of instruction, so this paper reports on an application of the theory, in this case the teaching of subtraction algorithms to Year 4 students, through the use of Multibased Arithmetic Blocks (MAB).

THE VIEW FROM COGNITIVE SCIENCE

Intelligent Tutoring Systems, typically costing hundreds of thousands of dollars to develop, have been shown to be an effective teaching medium in a range of areas (Polson and Richardson, 1988; Self, 1988). So why were students using an Intelligent Tutoring System to learn aspects of arithmetic not as successful as the developers of the system had hoped (Ohlsson, Bee and Zeller, 1990; Ohlsson, Nickolas and Bee, 1987)? Answers to this question include the need for the software to effectively represent or model mathematical concepts and skills, and for it to provide the opportunity for effective students interactions with the system.

Intelligent tutoring systems are part of the field of artificial intelligence, and their specification and development requires multidisciplinary teams to cover aspects of computer science and cognitive psychology. The combination of computer science, cognitive psychology and artificial intelligence forms the basis of the field of

Cognitive Science. In part cognitive science is concerned with declarative and procedural knowledge, with the use of analogy in learning, with machine learning, with planning nets and production rules that show the interconnectedness of learning, and with representations of the knowledge structure of the domain in question (Anderson, 1985; Holyoak and Thagard, 1989; Michalski, Carbonell and Mitchell, 1986). And it is these areas of study that provide the cognitive science basis for the procedural analogy theory discussed here.

THE VIEW FROM MATHEMATICS EDUCATION - BELIEFS

Concrete representation of mathematical ideas is hardly a new idea. Bowen (1972) suggests that as early as 3000BC the Mesopotamians were using a one-to-one correspondence to perform counting, and we are all familiar with tally sticks and with the abacus, though few of us may have actually used these. But how did concrete representation come to take on the importance it has today?

The movement away from book learning to a more pragmatic approach to education, to learning through experience, gained momentum from the seventeenth century through Bacon's work on empiricism, together with Descartes' writings on rationalism, Newton's scientific empiricism and Locke's idea of Tabula Rasa. These were followed by Rousseau's *Emile* in 1762, and the writings of Pestalozzi, Kant and Froebel, forming a body of philosophy expressing the importance of the individual, of the innate goodness of humanity, of the need to respect these qualities in education, and a view of children as more than simply 'little adults' (Bowen 1981; Boyd and King, 1966; Good and Teller, 1969; Mayer, 1973). These philosophies saw education as being child centred, and requiring the child to be an active learner. In the present century the work of Dewey, Montessori and the Progressive Education Movement have provided this child centred and active learning approach with further ideological support. Taking all these various components together, this body of knowledge provides a strong philosophical foundation for the value of concrete materials in teaching mathematics as a means of providing child centred, active learning.

But is this philosophical foundation sufficient reason for using concrete materials in contemporary mathematics education? Where else can support be found? In 1929 Durell suggested "the use of splits tied up in bundles of ten, the use of diagrams to illustrate fractions and the use of graphs" encouraged the saving of time, better retention and transfer of learning. Breslich (1933), Christofferson (1937) and Taylor (1938) also published articles supportive of the use of concrete materials with some of their ideas pre-empting contemporary approaches. So the use of concrete materials in teaching mathematics has been supported for decades in mathematics education literature. We can also cite the work of Piaget on concrete operational thinking and Dienes' writings on the structure of mathematics to support the use of concrete materials, and we can draw comfort from the position taken by the Nuffield Mathematics Project's slogan

I hear, and I forget
I see, and I remember
I do, and I understand.
(1967a,b)

But what are we expecting from the use of concrete materials? Exactly what kind of mathematical ideas are these materials supposed to represent, and what kind of learning is envisaged? What research indicates that concrete representations of mathematical concepts are actually valuable, that they do represent the concepts we intend, and that they do have a real and measurable impact on students' learning? And where is the data indicating exactly how concrete representations allow the learner to better arrange his or her cognitive structure so that learning is more effective?

What there are underlying ideologies and values about the roles of concrete materials, and that such beliefs may be as worthy of merit as empirical findings. It is possible to debate such values and beliefs, but data cannot be collected to prove that one view is more correct than another. These are issues fundamental to individual educator's beliefs about humanity, about students in particular, and what the individual sees as constituting teaching, learning and mathematics. That is, we use certain teaching approaches and materials in mathematics education because we believe in them. We may seek support for our opinions, but there is unlikely to be any logical necessity to accept or reject a particular point of view. Much of the pedagogy adopted by mathematics educators, and what mathematics educators tell other mathematics educators about effective pedagogies in research publications, curriculum development activities and teacher education courses is based on beliefs, with the addition of selected philosophical, anecdotal and empirical evidence.

THE VIEW FROM MATHEMATICS EDUCATION - RESEARCH

Resnick and Omanson (1987) sought to establish the relationship between performing arithmetic and understanding it, especially by illustrating procedural learning with "well-grounded mathematical principles". They developed a *mapping instruction* in which they maintained "a step-by-step correspondence between the blocks and written symbols throughout the problem". They had 80 fourth, fifth and sixth grade students perform tasks, both written and using MAB materials, where representations of numbers were constructed and decomposed, and where activities involved addition with carrying, and subtraction with decomposition. After a period of instruction, posttest scores showed children taught with the mapping instruction did not differ significantly from children in the comparison group, but in a delayed posttest the mapping instruction group gained higher scores. All the same, the researchers expressed disappointment at children's levels of achievement, and concluded that the mapping instruction was not effective in curing subtraction bugs. These findings suggest at least some of our beliefs about the value of concrete materials are questionable. Mathematics educators need to be concerned with these outcomes, especially since Resnick and Omanson's research was well designed, is frequently cited, was conducted by well known researchers and employed what appeared to be a detailed and sensible pedagogy - yet the use of concrete materials does not appear to have led to much in the way of positive outcomes.

In another important paper, Sowell (1989) reported a meta-analysis of 60 studies designed to assess the value of manipulative materials in mathematics instruction. The studies ranged from those involving kindergarten children to those in which college students participated, and employed a wide range of manipulatives and mathematics topics. Sowell found that treatment lasting a school year or longer favoured the manipulative groups but only for the use of concrete materials and not for pictorial representations. Treatments for shorter periods showed no difference between the manipulative and nonmanipulative groups on either posttest or delayed posttest scores.

Reference to a wider range of research literature simply confirms the uncertain value of concrete materials (for example, Hart, 1989; Hiebert and Carpenter, 1992). Treatment time is an element in teaching and learning, and so is pedagogy, but one of the difficulties in analysing the literature on concrete materials is the lack of detail given about the actual teaching methods employed. Statements about an experimental teaching approach contrasted with a traditional approach give insufficient detail as to the intricacies and nuances of the learners' experiences. I believe it likely that the pedagogy used in some studies could be improved simply through more attention to detail in the teaching learning process. For example, the procedural analogy theory outlined below provides one set of guidelines for improving instruction.

THE PROCEDURAL ANALOGY THEORY

How then can we show that the construction of meaning, through the internalisation of mathematical skills and concepts into a richly connected cognitive network, will be assisted by the manipulation of concrete materials? If teaching is to lead to understanding and to the learning of those standardised written procedures that continue to be an important goal of school mathematics such manipulation will need to be complemented by a pedagogy that encourages cognitive re-construction.

The procedural analogy theory describes how concrete materials assist the learning of declarative and procedural knowledge, and movement to the required target behaviour. Simplification, procedural analogy and symbolism, together with practice, lead finally to automatic responses. The procedural analogy theory is a theory of instruction, and has its basis in both cognitive science and mathematics education. In addition to the original publication concerning this theory (Ohlsson and Hall, 1990), aspects of the theory have been presented elsewhere (Hall, 1990, 1991, 1992a, 1992b). In solving a particular problem the theory relies heavily on the analogy between the process of acting upon concrete materials representing mathematical concepts and skills, and the written algorithm that corresponds to that process. The theory asserts that while concrete materials may be used in a wide range of ways to achieve a correct answer, there are some ways that will be more effective than others because they more closely mirror the desired target behaviour, and these latter ways provide the more effective teaching approach.

Table 1 shows one use of MAB materials and the target algorithm that is developed from this material. The steps emphasised both in the use of MAB materials and in the target algorithm are not unique, and must be

Table 1: Procedural analogy: MAB and target procedures

MAB procedure	Target procedure
0.0 428 - 169	0.0 428 - 169
0.1 Subtract 169 from 4H, 2T, 8U	
1.0 Process units	1.0 Process units
1.1 Take 9U from 8U (cannot)	1.1 Take 9 from 8 (cannot)
1.1.1 Trade for more units	1.1.1 Trade for more units
1.1.2 Move 1T from 2T to bank, bring back 10U	1.1.2 Recall 2 - 1 = 1
	1.1.3 Cross out 2, write 1
1.1.3 Join 10U and 8U	1.1.4 Write 1 next to 8
1.1.4 Recall 10U + 8U = 18U	1.1.5 Recall this is 18
1.2 Take 9U from 18U	1.2 Take 9 from 18
1.3 Recall 18U - 9U = 9U	1.3 Recall 18 - 9 = 9
1.4 Record answer, 9U in answer space	1.4 Record 9 in answer space
2.0 Process tens	2.0 Process tens
2.1 Take 6T from 1T (cannot)	2.1 Take 6 from 1 (cannot)
2.1.1 Trade for more tens	2.1.1 Trade for more tens
2.1.2 Move 1H from 4H to bank, bring back 10T	2.1.2 Recall 4 - 1 = 3
	2.1.3 Cross out 4, write 3
2.1.3 Join 10T and 1T	2.1.4 Write 1 next to 1
2.1.4 Recall 10T + 1T = 11T	2.1.5 Recall this is 11
2.2 Take 6T from 11T	2.2 Take 6 from 11
2.3 Recall 11T - 6T = 5T	2.3 Recall 11 - 6 = 5
2.4 Record answer, 5T in answer space	2.4 Record 5 in answer space
3.0 Process hundreds	3.0 Process hundreds
3.1 Take 1H from 3H	3.1 Take 1 from 3
3.2 Recall 3H - 1H = 2H	3.2 Recall 3 - 1 = 2
3.3 Record answer, 2H in answer space	3.3 Record 2 in answer space
4.0 Read answer (2H 5T 9U)	4.0 Read answer (259)

developed by the teacher. Once the teacher has decided on the target behaviour, a teaching sequence can be developed for the concrete materials that increases the likelihood that learners will structure their own knowledge in a similar manner.

The procedural analogy theory uses an isomorphism index ($I_{1,2}$) as a measure of analogy between the two procedures. The index is given by the formula

$$I_{1,2} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

where N_1 is the number of steps in the first procedure, N_2 the number of steps in the second procedure, D_1 the number of steps in the first procedure but not in the second, and D_2 the number in the second procedure but not in the first. In Table 1, $N_1 = 25$, $N_2 = 26$, $D_1 = 3$ and $D_2 = 4$ giving a high isomorphism index of 0.86. Slight variations in the steps will lead to a lower isomorphism index. The theory argues that the closer the relationship between the procedure involving the use of concrete materials and the target procedure, the higher the $I_{1,2}$ value, so the more effective will be the value of the concrete materials, and the greater the level of learning outcomes. That is, the procedural analogy theory allows an analysis of teaching steps prior to teaching and provides a method of measuring likely pedagogical success.

METHODOLOGY

The research reported here involved 110 students, two Year 4 classes in each of two schools, where students were randomly assigned to one of three groups to learn subtraction algorithms through the use of MAB materials. For each school the regular classroom teachers taught one group during the period of the research, the researcher taught the third group. Two of the three groups used a teaching method where there was a high isomorphism index (High I), one group using expanded numerals in the movement from concrete materials to written algorithm, the other moving directly from concrete materials to the target procedure. The third group used a method with a lower value isomorphism index (Low I), where the teaching approach was acceptable to all teachers in the research, was a typical teaching approach and one supported by curriculum statements and textbooks. These details are summarised in Table 2.

Table 2: Teaching groups

School	Group/Teaching Approach	Teacher
A	High I, expanded	1
	High I	2
	Low I	3
B	High I, expanded	4
	High I	5
	Low I	2

The teaching approaches of the two experimental groups differed from the teaching approach of the comparison group in the detail of the correspondence between the actions on concrete materials and the written algorithm, and in the detail of the guidance and description given by the teacher.

RESULTS AND ANALYSIS

All students were given a pretest on subtraction algorithms, and parallel posttest and delayed posttest. An analysis of variance showed no significant differences in pretest scores between students in the three teaching approaches. An analysis of variance on posttest scores showed there was a trend for students in the High I teaching approaches to have higher scores than for students in the Low I approach, but the differences were not statistically significant ($p < .10$). An analysis of variance on delayed posttest scores showed a significant difference ($p < .05$) in favour of the High I teaching approaches over the Low I teaching approach. There was no statistical difference in scores between members of the two groups with High I teaching approaches.

The trend for students exposed to teaching approaches with high isomorphism indices to show greater gains on posttests scores than students where the teaching approach had a low isomorphism index is a positive finding in terms of the procedural analogy theory, but clearly needs further investigation. It may be that for a range of teaching approaches students will not score significantly differently on posttests if the test is administered immediately after completion of the topic, and that differences in test scores resulting from different teaching approaches become evident only over time after instruction has finished. The topic chosen may also explain the sameness of the posttest scores. That is, subtraction was not a new topic for any of the students involved in this study, so posttest results may have differed had the topic been new to all students.

The finding that students exposed to teaching approaches with high isomorphism indices showed greater gains on delayed posttest scores than those students where the teaching approach had a low isomorphism index gives some support to the procedural analogy theory and to the longer term value of using concrete materials. These findings also suggest that the use of concrete materials together with specific pedagogies assist the development of an effective cognitive structure, one in which concepts and skills are stored in a meaningful and efficient manner, where they can be remembered, recalled and reconstructed as necessary.

DISCUSSION

How generalisable is the procedural analogy theory in terms of mathematics topics and ages of learners? Does application of this procedural analogy theory encourage learners to develop a richly connected network of cognitive structures? Is the cognitive development taking place through the application of this procedural analogy theory superior to developments using other teaching approaches?

This research has generated many questions, there is clearly a need for the research to be replicated, and further investigation is necessary to assess the relevance of the procedural analogy theory in other school mathematics topics. At the same time the research reported here appears to have some potential in facilitating the design of teaching approaches involving the use of concrete materials, and in helping teachers guide learners in the construction of knowledge.

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