THE CONCEPT OF FAIRNESS IN SIMPLE CAMES OF CHANCE

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This study examines the mathematical concepts of "fairness" and "expectation" in probabilistic situations. The subjects were 40 high school students in Semester I, Year 11, Maths in Society classes in three Queensland high schools. Twenty "gamblers" were identified by questionnaire and subsequent interview. A control group of similarly achieving "non-gamblers" was selected. The research compares the ability of each group to contract a working definition of the concept of mathematical expectation and to use this concept in determining the fairness of a number of games of chance.

This study examines the mathematical concept of "fairness" as it applies to simple single-event games of chance involving coins, dice and cards.

In the determination of fairness, two aspects are examined:

-the misuse of an heuristic of "representativeness" or "availability"

-the use of an intuitive understanding of the concept of "expectation".

Misconceptions in probabalistic reasoning involving the use of "representativeness" and "availability" heuristics have been well documented by researchers including Shaughnessey (1977, 1981, 1983), Scholtz (1986), Tversky and Kahneman (1982), and Peard (1991a, 1991b, 1991c).

The use of "representativeness" to determine the fairness of a coin or game is illustrated when in situations the subject takes a short term sequence of events as being "representative" of the long term situation and erroneously concludes bais or unfairness.

"Availability" is used to come to the same conclusion by reasoning that such short term sequences are not readily recalled. More "balanced" results are more readily "available".

Bright, Harvey and Wheeler (1981) in a study of fair and unfair games claim that "fairness" is best described by calling attention to an intuitive understanding of "unfairness". In refering to students in years 4-8 they claim that "Helping students recognize when a situation is fair or unfair is a reasonable expectation of the school curriculum." (p.50). Research by Anderson and Pegg (1988) also reported difficulties primary school pupils encountered with the determination of fairness.

The mathematical concept of fairness, as opposed to a merely intuitive understanding, relies on the concept of "expectation". A game is "fair" if all participants have equal mathematical expectation. This in turn requires an understanding of mathematical expectation which is defined as the product of probability and return.

These concepts are clearly beyond the elementary level but require the application of only basic probabilistic reasoning. For simple games involving only two players, one need only determine the probabilities for each to win and then calculate the required amounts for each to be a constant product (or inverse proportion). This constitutes an effective concept of equal mathematical expectation for both players.

Bright et. al. note:

In complex situations it may be difficult to determine mathematically whether a situation is fair. (p 50)

Lovitt and Clark (1988) questioned whether pupils about to leave school had realistic ideas about the outcomes of gambling and concluded that "there is a huge gap between perception and reality" (p.77) in which pupils demonstrated misconceptions of the concept of expectation. Although they did not refer to any heuristic involved in arriving at these misconceptions, it would appear that an "availability" heuristic was in fact involved.

The inclusion of basic probability and its applications in the general school mathematics curriculum, both elementary and secondary, has been a relatively recent development. Pereria and Swift (1981) writing in the N.C.T.M. Yearbook made a strong arguement for probability to be part of every students education. Since then considerable progress has been made world wide as is evidenced by the N.C.T.M. statement of Standards in the

U.S.A. and the inclusion of "Chance and Data " in the Australian National Statement which makes specific reference to "fairness" and "expectation".

In Band B (upper primary) only an intuitive notion of fairness is expected. Possible activities include:

"Make non-numerical predictions about about equally likely events such as those involved in rolling a fair die and compare predictions with results of experiments. (p.170)

Reference to expectation first occurs in Band C (lower secondary) where possible activities include:

"Investigate uses of probability in insurance... Study common games of chance to find the expected return....note that statements of odds which appear in gambling contexts reflect statements of subjective probability as well as statements of return on money invested....return on a win may be high but the chance of that win is correspondingly small." (p.175).

In Band D (upper secondary) possible activities include:

"Devise, play and analyse a variety of "fair" and "unfair" games. Calculate and interpret expected values..."(p.182).

However numerous difficulties with the implementation of such programs have been reported. In Australia, teacher unfamilarity with much of the content is recognised. See for example Peard (1987). Pedagogical problems with the teaching of probability are also well documented. See
Garfield and Ahlgren (1986, 1988), Kapadia (1984), Brown (1988), Pegg (1988), Green (1982, 1986), del Mas and Bart (1989).

Thus it is reasonable to assume that at the present time very few students will have had formal instruction in the topics of fairness and expectation prior to the Senior Secondary grades and that only some will gain this knowledge in these years.

OBJECTIVES:

The subjects in this study were 40 high school students in Semester I, Year 11, Maths in Society classes in three Queensland high schools. Two of these schools were in a lower socio-economic region, close to horse racing,dog racing and trotting tracks. Many senior students in these schools followed the races.

The study is part of a larger study investigating the construction of various probabilistic concepts within a social context by students whose background includes a familiarity with the phenonomen of gambling, particularly in relation to "track" betting. These are subsequently refered to as "gamblers". Interview questions established that all of these subjects were familiar with betting in track situations, the use of "odds", and methods of calculating payouts.

The objectives of the study were to determine:

(1) The pupils ability to recognise fairness in simple games of chance.

(2) Whether or not an heuristic was misused in incorrect identification.

(3) Whether or not there was any difference in this ability between "gamblers" and "non-gamblers"

(4) The ability of the students to recognise or construct a concept of expectation in simple games of chance in which players have unequal chances.

(5) The ability to use the concept of expectation in determining fairness.

(6) Whether or not these abilities were related to: social background (gambling), school achievement, gender.

METHODOLOGY:

The "gamblers" were identified by questionnaire administered with the help of either the regular classroom teacher or a special needs teacher. A subsequent interview was given to validate responses. Only those indicating a "great deal" of interest in at least one form of track racing were considered as "gamblers". A control group of "non-gamblers" was selected from those responding negatively to all forms of gambling and games of chance.

All schools were coeducational and an approximately equal number of male and females responded positively to interest in gambling. Thus a balance of subjects by gender was easily obtained. A balance of subjects by achievement was also obtained.

The research methodology employed was that of the structured clinical interview as described by Romberg and Uprichard (1977).

The interview asked open-ended questions relating to:

Category 1 - Representativeness and Fairness.

- the subjects' ability to recognise when a simple game of chance is "fair" and whether or not a heuristic of representativeness or availability was used in the decision making.

Questions:

The first questions asked were of the type:

1. (a) "You and I play a game of chance in which a coin is tossed. Heads I win, Tails you win. Of the last 15 people who played this game with me 10 lost. Is this a fair game?"

Similar questions relating to rolling a single die and drawing cards from a deck followed.

These questions are similar to those asked by Shaughnessey (1981). He reported a high incidence of the use of availability to conclude that the coin tossing game was not fair.

Those believing the games to be unfair do so by either using the short term results, for example, of 15 tosses to be "representative" of the long term probability of the coin or reply that they expect the next person to lose since "people tend to lose at this type of game" (availability).

Thus the next questions asked in this study were:

(b) "Is the coin/die/card game fair?"

(c) " Why?" or "Why not ?", depending on response.

Follow-up questions in the structured interview were of the type:

To those who responded affirmatively to (b)

(d) "How many tosses would you need to conclude that the coin was unfair?"

Those who recognised that a very long run was required before bais could be suspected were considered to be free of the misuse of the representativeness heuristic.

2. (a) "You and I play a game of chance which involves throwing a single die. We each bet $1, winner takes the $2. If the numbers are 1, 2, 3 I win, if they are 4, 5, 6 you win. Is this a fair game?"

(b) "If we change the rules so that if they are 1, 2, 3, 4 I win, 5, 6, you win. Is this a fair game now?"

(c) "Why or why not?"

(d) "Can we change the amounts each player puts in to make this game fair?"

This last question then leads in to the concept of "expectation"

Category 2 - Expectation and Fairness.

Questions: (following from above)

3. (a) "Since I have the better chance of winning can we make the game fair by increasing the amount I put in?"

Those who responded negatively to this were considered to have no concept of expectation. Typical responses were:

" You will still have a better chance than me and that's not a fair game."

To those who responded affirmatively:

(b) "How much should I put in?"

To demonstrate a basic understanding that expectations can be made equal, it was not required that the subject use formal mathematical language. A typical response was:

"Well you have four chances to my two, that's twice as many. So if you put in twice as much, that would

be fair."

The extent of understanding was investigated further:

(c) "What if I chose five numbers and left you with only one? How much should I put in now?"

For those who were able to answer this correctly different situations were then investigated.

e.g.

(d) "If we draw cards from a deck and I choose any Ace leaving you the rest, how much more than me should you put in to make the game fair?"

(e) "If I choose just one card such as the Ace of Spades,how much now?"

(f) "If I choose the 16 "coloured cards" - ace, king ,queen,jack of each suit, leaving you the 36 remaining cards and I put in $1, how much should you put in?"

Those who were able to demonstrate consistently in all of these situations that "fairness" can be established by each contributing an amount in inverse relationship to the probability (i.e. an equal product of probability and return or equal expectation) were considered to have a complete understanding of the basic concept.

An exact answer to the last question was required for this. It was not sufficient to reason along the lines (as did some):

"I have more than twice your chances so I should put in more than twice as much."

A "complete" understanding required reasoning that resulted in the calculation of 36/16 x $1 = $2.25.

RESULTS AND ANALYSIS OF DATA:

From the responses to these questions subjects were classified:

Category 1 - Representativeness and Fairness

(1) Recognises a fair simple game

33 of the 40 were able to recognise that in all situations the game/coin/die were fair and that deviations were not unreasonable.

(2) Uses an heuristic to misjudge a fair game

5 of the 40 were classified in this category.

Of these 3 responded using the "representativeness" heuristic and 2 using an "availability" heuristic.

2 reponded that they were unable to make a decision.

None of the 5 used the heuristic in questions of the type of 1(e),(f)-very short sequences.

5 of the 7 were non-gamblers but due to the small size of this category no test of significance was performed.

Rather, we note that the majority of both gamblers and non-gamblers were able to recognise that the situation itself was in fact fair.

(3) Free of the "representativeness" misconception

(correct response to Q.1(d))

Of the 33 who recognised fairness 23 were able to conclude correctly that a much longer sequence than that given would be required to infer bias or unfairness.The others were unsure or undecided.

Category 2 - Expectation and Fairness

(1) No knowledge of mathematical expectation.

These subjects were unable to answer Q3(a) correctly and would tend to reason: "A game can only be fair if each player has the same chance of winning" Two "non-gamblers" admitted to having no basis on which to make decisions of fairness.

Total: 21 Gamblers : 8 Non gamblers: 13

(2) Some intuitive knowledge of the use of expectation in determining fairness. These subjests answered questions 3 (b) and (c) correctly but were unable to answer all of the more complex questions 3 (d) - (f)

Total: 13 Gamblers :10 Non gamblers: 3

(3) A thorough knowledge of the basic concept of mathematical expectation as demonstrated by their responses to all parts of question 3
Total: 6 Gamblers : 5 Non gamblers: 1

The Null hypothesis

Ho: "There is no difference between the gamblers and the non-gamblers in their knowledge of mathematical expectation." was tested using a Chi-squared test of statistical significance and rejected at the 5% level.

Table 1

	Observed				Expected Under Ho			
	None	Some	Thorough		None	Some	Thorough	
Non-Gamblers	13	3	1	18	9.45	5.85	2.7	18
Gamblers	8	10	5	22	11.55	7.15	3.3	22
Total	21	13	6	40	21	13	6	40

An analysis of this ability by achievement:

Table 2

	Observed			
	None	Some	Thorough	
HA	8	8	3	19
LA	13	5	3	21
Total	21	13	6	40

Table 3

	Oberved			
	None	Some	Thorough	
Male	8	7	3	18
Female	13	6	3	22
Total	21	13	6	40

In both of these the Null hypothesis cannot be rejected and we conclude that this knowledge is not related to either school achievement or gender.

IMPLICATIONS:

Category 1

Since the misuse of an heuristic to conclude unfairness was not common amongst either group we cannot compare groups. These misconceptions were not as frequent as is reported in the literature. Shaughnessey (1981), for example, found the misuse of availability to imply unfairness widespread even amongst college entrants. Tversky and Kahneman (1982) noted that "misconceptions are not limited to naive subjects" (p.5). However Kapadai (1984) has questioned much of this research and suggests that some of the misconceptions may actually refer to misinterpretation of the question. The results of this study which were obtained from a structured clinical interview rather than questionnaire or test items would seem to support Kapadia in this.

Category 2

The fact that the gamblers were significantly better at using expectation to determine fairness has a number of important implications.

First, the concept is not part of the regular school curriculum - they do not use the term "expectation" but construct what is essentially an equivalent procedure.

Or as Davis (1989) says "as intelligent responses to their environment"(p.32).

Second, since all of the gamblers were familiar with track betting, the use of "odds" in betting situations and the calculation of resulting payouts, it is hypothesised from the results of this study that this mathematical knowledge may be attributed to the prevalence of gambling within the social background of this group.

The fact that this ability did not relate to school achievement or gender would tend to give support to the hypothesis.

As such, the knowledge may be considered as a form of "ethnomathematics" as defined by D'Ambrosio (1985):

> *..mathematics which is practised among identifiable cultural groups (whose) identity depends largely on focuses of interest and motivation. (p. 45)*

This has implications for the classroom teacher. As Clements (1988) says "It needs to be remembered that often in Australia there are unique factors influencing how children learn mathematics." (p.5)

With the concepts of fairness and expectation now specifically within the curriculum, the teacher must be aware of the knowledge that pupils bring with them to the classroom.

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