NEGATIVE NUMBERS; CONCRETE AND FORMAL IN CONFLICT?

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Provisional results are reported here from developmental research on negative numbers in the upper grades of primary school. An attempt has been made to deal with learning strands both on concrete magnitudes or quantities and on formal constructs. the observation of individual learning process is the core of the research. The results show that - in the learners- a gradual shift takes place either from the realistic context to the mathematical one or from the known mathematical context to the new one.

Negative numbers occurred differently in the history of mathematics. On the one hand they were connected with concrete magnitudes and quantities (e.g. possesses and debt)[1] and on the other hand with by-products of, or constructs to support the execution of algebraic operations (Fischbein, 1987).

This means that negative numbers appeared both as concrete numbers and as formal mathematical constructs. Both manifestations provoked cognitive conflicts in mathematicians for centuries (Glaeser, 1981)[2].

For the formal one no intuitions, to use Fischbein's terms (1987) can be developed (or mental objects as Freudenthal (1983) would have called them).

Moreover the existence of concrete manifestations of negative numbers blocked the acceptance of negative numbers as meaningful mathematical through-things (or nooemena according to freudenthal (1983)). And 'less than nothing','$-3 <$ but $(-3)^2 > (2)^2$' are controversial expressions indeed from the perspective of concrete magnitudes or quantities.

It was Hankel who solved the problem of controversion in negative numbers definitely in 1867. In his 'Theory of complex numbers' he formulated his principle of permanence. This means that negative numbers ass an operational structure had to fulfil the demand of algebraic consistency, that is obeying the law of distributivity of multiplication over addition.

E.g. Distributive law: $(a + b)(c+d) = ac + ad + bc + bd$, thus

$$(4 + {}^-3)(5 + {}^-4) = 4 \text{x} 5 + 4 \text{x} {}^-4 + {}^-3 \text{x} 5 + {}^-3 \text{x} {}^-4 = ?$$

$4 \text{x} {}^-4$ permits a concrete interpretation (that means intuitive in Fischbein's terms or at the level of mental objects in Freudenthal's terms), namely '4 times a debt of 4 (...)'. ${}^-3 \text{x} 5$ can only be interpreted in the same manner it is replaced by $5 \text{x} {}^-3$ due to the law of commutativity and that is a formal, mathematical operation! Finally ${}^-3 \text{x} {}^-4$ needs to be interpreted as ${}^+12$ in order to achieve the same outcome as $(4 + {}^-3)(5 + {}^-4)$ has.

So, according to Hankel's view, negative numbers are after all formal mathematical constructs.

THE TEACHING OF NEGATIVE NUMBERS

The historical dilemma as described briefly, has its educational counterpart. One the one hand one cannot avoid 'possess and debt', 'above and below sealevel' and so on, but at the same time it will be impossible to neglect the formal arguments to make the negative numbers 'full-grown' mathematically. This means that a course on negative numbers also needs to contain a learning strand aiming at formalisation, which means algebraising the structure of negative numbers (by-product approach).

Because of the cognitive controversion between the to manifestations of negative numbers and in order to avoid misconceptions in the learner because of false concretisations one also might choose for a by-product course only (cf. Liebeck, 1990, p. 238).

An important question is now: Will such a course with learning strands on both concrete magnitudes or quantities and by-products be possible? And derived from that: What additional features should the by-products strand necessarily have?

THE PRESENT RESEARCH

Kind

The research to be reported here is exploratory, developmental research (cf. Freudenthal, 1991, and Streefland, 1991). Such research aims both at the development of a prototype of a course (on negative numbers in the present case) and at bringing forth theory regarding the teaching and learning of the subject (this also includes extending the existing theory within the same general framework for mathematics education).

The Course

What will be reported here is based on a course consisting of two parts. Part I: Events at the busstop. The changes in occupation of the bus on ride, can be described by means of events, either expressed in number pairs of passengers getting on and off the bus, for instance (5, 10) or in single numbers expressing the change, (5), in the number of passengers. Part II: Column subtraction from left to right, for instance:

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324
258-
134
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This way of calculation can evoke the need for the construction of the learners' personal notation of the shortages. Will it be possible to subtract also numbers in shortage-notation? What happens in the hundreds', tens' and units' position, how can it be related to busstop-events? And what is the role of numbers and their opposites?

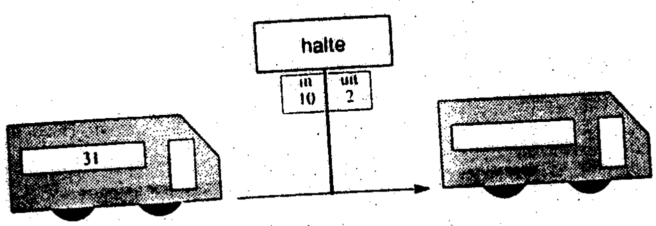
N.B. Different contexts will be explored later too, like 'temperature' and 'dry and wet numbers in locks'.

Additional Features of the Course

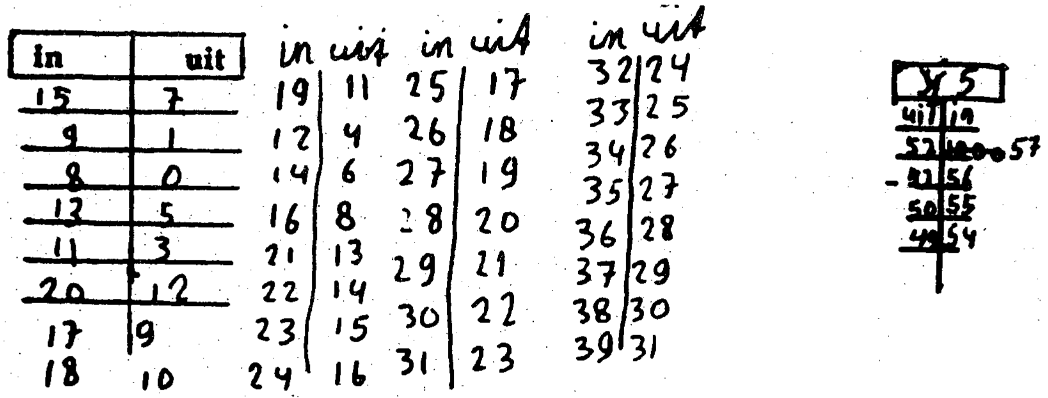
As the first example shows, the number pair appearance of negative numbers can be met easily; one single negative number 'hides' and (infinite) set of number pairs.

The second example implies that the learners can apply their known way of subtraction to find out how they should operate with shortages. Their normal procedures wither hands-on or with calculators act both as a framework for *making plausible* and for *verification*.

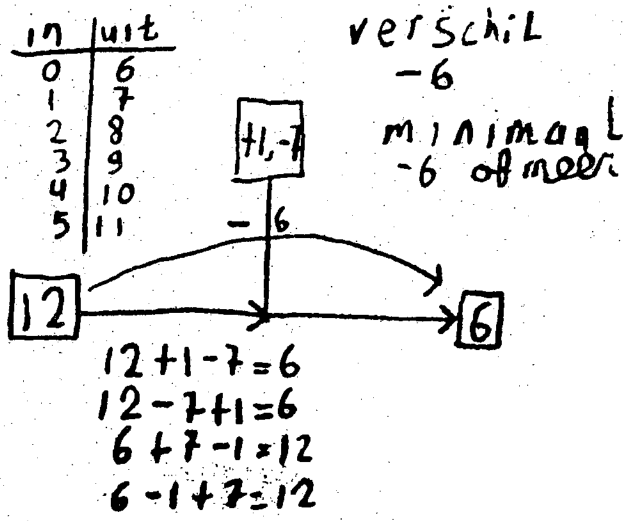
Problem:



31 people on the bus; 10 on and 2 off at the stop; how many.........?



Paradigmatic examples of pupil's work



The following characteristics draw attention:

- treating busstop-events from a change-in-occupation point of view;
- filling in busstop-tables either arbitrarily or systematically;
- shifting from filling in arbitrarily to systematically;
- meeting restrictions set by the context, for instance either starting or ending with the maximal numbers of passengers getting on and off the bus that are possible;
- (de-) contextualising and algebraising (cf. the number sentences).

The Group

The group (17) consists of the two combined uppergrades of a primary school (11-12 years) and is of average ability.

Methodology and Methods

In general the observed learning processes are considered to be the starting point for the determination of 'How to proceed?' As a consequence the question is not how to achieve a set of predetermined goals for negative numbers, operations included, but rather to find out how the learners' notions on negative numbers develop.

Therefore the observation of the teaching and the (individual) learning processes are at the core of the researcher's activities.

The ongoing activities consist of a teaching experiment intervened by individual interviews.

PROVISIONAL RESULTS

The results will be considered in relation to the aims of developmental research (prototype and theory). Since the results are still provisional (the research has not been finished yet), only some essential features can be touched upon here.

Individual Learning Processes

Part I: Events at the busstop.

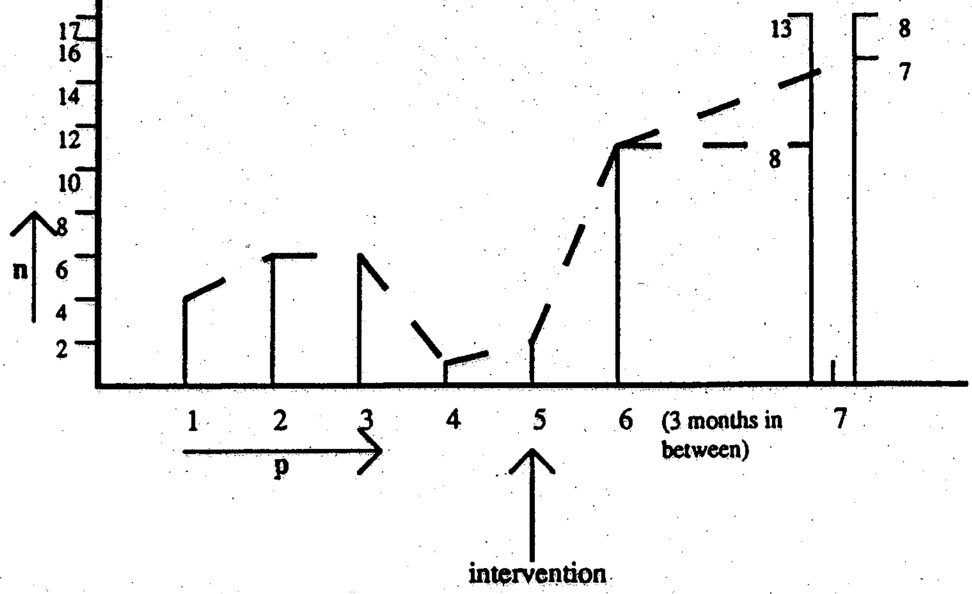
Different aspects of the educational context for this part of the course were:

- collaboration in practical work (worksheets) in small groups;
- interactive sessions on the constructions and productions by the learners which means discussing and negotiating them;
- producing problems of busstop events freely by way of assessment.

Interventions were made by means of (clerical) interviews. The buses 'drove during four lessons'. The assessment took place three months afterwards. The different aspects will be shown in an example of students work first. The learning history of the complete group will be considered from a double perspective, namely:

- the presence of (non)-systematic tables (1), and
- (de-)contextualisation and algebraisation (2).

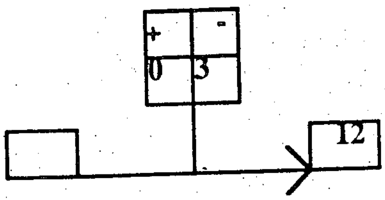
(1): For six problems the pupils filled in tables. The fifth problem was discussed in an interactive session. It was then that filling in systematically was discussed and reflected upon for the very first time with the group as a whole. The following graph shows that working systematically improved considerably after the interaction.



Intervention by interactive session; assessment after three months; n = number of pupils, p = problem number; problem 7 was a free production assessment task concerning the invention of bus problems with tables and number sentences.

It came out from the interviews after the assessment that all the pupils who did their tables systematically party in advance of the interactive session said that they would have used an eventual second opportunity to change their non-systematical tables into sysematical ones. The aim was to develop the skills of connecting number pairs and their (negative) outcomes.

(2): 'Having the bus driving backwards' (db) was invented spontaneously by 8 out of 17 pupils for the following problem:



Two of them were able to make fitting number sentences like 12 + 3 - 0 = 15.

The challenge of the assignment of the assessment provoked the production of varying number sentences according to features like (de)-contextualisation and algebraisation. The table on the next page contains the results. Three pupils even formulated rules like:

'If you calculate the other way round, that is from left to right, the pluses become minuses and the minuses pluses.'

task	db	alg
1	10	7
2	5	2

Number sentences (de)-contextualised and algrebraised

N.B. This concerns a free production task again. The fall back in registered results can be explained by the kind of assignment, the time factor and ciromstances like that.

Part II: Column subtraction from Left to Right.
Here it turned out that the problems were counter-intuitive as soon as an overall result was negative. For instance:

1823
-9745
8122

"This one is wrong because there is nothing left from which you can subtract your debts. So I have a debt of 7922
no debt - 100
So I have a debt of 7822

As long as the shortages were not in the place with the largest value the operations performed by the pupils went well in general. 13 out of 17 pupils applied the notation with shortages in the assessment task, while the remainder fell back to their usual method. The rule was formulated that two shortages of the same 'size' have a difference of zero.

DISCUSSION

The research showed thusfar that the historical conflict of the two manifestations of negative numbers is also met by the pupils of the group that participates in the experiment. Moreover it came out that the context of the busstop provides for the possibility to notate number sentences in the opposite manner. This can be considered as a prestage to notating them formally, or to put it differently: It may be hypothesised that between the positive and the negative numbers there - from the learners point of view - seems to be an interjacent area where negative numbers occur implicitly. It will be our further aim to chart this area.

Finally it came out that the learning processes can be characterised by shifting gradually from a concrete to a partly formalised level. This shifting process consists of different, subsequent shifts, that are provolked by reflections in the learner on what is going on.

Notes

1. The names for negative numbers used by Brahmagupta (about 600 a.C) referred to debt and property.
2. Lazare Carnot (1753-1823), writes, referring to operations with negative number: "A multitude of paradoxes and relative absurdities result from the same notion; for instance, that -3 would be smaller than 2; nevertheless $(-3)^2$ would be greater than $(2)^2$, that it to say that, considering two unequal quantities, the square of the greater would be smaller than the square of the smaller: This shocks every clear idea one may get, referring to the notion of quantity" (quoted from Fischbein, 1987, l.c. p. 99).

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