REASONING BY ANALOGY IN CONSTRUCTING MATHEMATICAL IDEAS

Lyn D. English

Centre for Mathematics and Science Education

Queensland University of Technology

ABSTRACT

Analogy appears to be one of the most important mechanisms underlying human thought, at least from the age of about one year. A powerful way of understanding something new is by analogy with something which is known. The research community has given considerable attention to analogical reasoning in the learning of science and in general problem solving, particularly as it enhances transfer of knowledge structures. Little work, however, has been directed towards its role in children's learning of basic mathematical ideas. This paper examines analogy as a general model of reasoning and proposes a number of principles for learning by analogy. Examples of analogical reasoning in children's mathematical learning are presented, including children's ability to recognize similarity in problem structure which was investigated in a recent two-year study. The proposed principles are applied to a critical review of some commonly used concrete analogs and to a brief analysis of more abstract analogs, namely, established mental models which serve as the source for the construction of new mathematical ideas.

INTRODUCTION

It has been argued that much of human inference is basically analogical and is performed by using schemas from everyday life as analogs (Gentner, 1989; Halford, 1992). Given that analogy is a natural and ubiquitous aspect of human cognition, analogical reasoning would seem to lie at the very core of our cognitive processes. It is even used by very young children under appropriate conditions (e.g., Gholson, Dattel, Morgan, & Eymard, 1989; Goswami, 1991). Such reasoning is also responsible for much of the power, flexibility, and creativity of our thought (Halford & Wilson, 1993; Holyoak & Thagard, 1994).

In 1954, Polya devoted an entire volume to the use of analogy and induction in mathematics. While he demonstrated how analogies can provide a fertile source of new problems and can enhance problem-solving performance, his ideas were not as widely adopted as some of his other work, largely because they were descriptive rather than prescriptive (Schoenfeld, 1992). More recent studies however, have given greater attention to analogical reasoning in general problem solving, particularly as it enhances transfer of knowledge structures (e.g., Holyoak & Koh, 1987; Novick, 1990, 1992). While research has also addressed the role of analogy in science learning (e.g., Clement, 1993; Duit, 1991; Stavy & Tirosh, 1993), little work has been directed towards its role in children's learning of basic mathematical concepts and procedures. The purpose of this paper is to examine analogy as a general model of reasoning and to highlight its role in children's elementary mathematical learning.

NATURE OF ANALOGICAL REASONING

A commonly cited definition of analogy is that of Gentner (1983, 1989), namely, an analogy is a mapping from one structure, the base or source, to another structure, the target. The system of relations that holds among the base elements also holds among the target elements. Normally the source is the part that is already known, whereas the target is the part that has to be inferred or discovered. An analogy utilizes information stored in memory (Halford, 1992). In this way, a model of analogical reasoning shares common features with knowledge-based models of reasoning (e.g., Chi & Ceci, 1987). However analogies go beyond the information retrieved because the interaction of the base and the target produces a new structure that extends beyond previous experience (Halford, 1992). Furthermore, employing an analogy can open up new perspectives for both perceiving and restructuring the analog (Duit, 1991). The acquisition of this new structure is in accord with the constructivist views of children's learning; that is, learning is an active construction process that is only possible on the basis of previously acquired knowledge (Davis, Maher, & Noddings, 1990; Duit, 1991). In other words, learning is fundamentally concerned with constructing similarities between new and existing ideas.

ANALOGICAL REASONING IN CHILDREN'S PROBLEM SOLVING

Analogical reasoning plays a significant role in problem solving. The ability to utilize a known problem (i.e., a base or source problem) that has an identical goal structure to the new problem to be solved (target problem) can enhance problem-solving performance (e.g., Novick & Holyoak, 1991). This analogical transfer involves constructing a mapping between elements in the base and target problems, and adapting the solution model from the base problem to fit the requirements of the target problem (Novick, 1992).

In a recent study (English, 1994), 9 to 12 year-olds from low, average, and high achievement levels in school mathematics were individually administered sets of novel combinatorial and deductive reasoning problems presented in concrete and isomorphic written formats. The concrete combinatorial problems involved dressing toy bears in all possible combinations of colored T-shirts, pants, and tennis rackets. The number of combinations ranged from 9 to 12. The isomorphic written examples required the child to form all possible combinations of: a) colored buckets and spades, b) colored shirts, skirts, and shoes, and c) greeting cards featuring different colors, lettering, and messages. The hands-on deductive problems entailed working through a series of clues to determine how to: a) arrange a set of playing cards, b) stack a set of colored bricks, and c) match names to a set of toy animals. In the isomorphic written examples the child used given clues to determine: a) the locations of families in a street of houses, b) the location of a particular book in a stack of books, and c) the identification of personnel who played particular sports.

Upon completion of each of the sets of combinatorial and deductive problems, children were asked whether solving one set (either hands-on or written) assisted them in solving the other set. Children were also asked if they could see ways in which the problem sets were similar. Results to date indicate that, on the whole, the older children were better able to identify the structural similarities between the problems than the younger children. There were however, several cases in which the younger children performed better than their older counterparts in recognizing these similarities. This was also the case for children in the lower achievement levels who often performed just as well, if not better, than the high achievers. For example, 9 year-old Hayley, a low achiever, stated that the sets of combinatorial problems were similar because "you have to use combinations ... you have to do them in a method so you don't get get two exactly the same." On the other hand, Nicholas, a high-achieving 9 year-old, commented that the problems were "about dressing ... about matching colors." The older children frequently made mention of the similarity in the number of sets that had to be matched. For example, 12 year-old Natalie commented that the final two written problems (of the form, X x Y x Z) were like the final two hands-on examples because they had "three things to match up."

For the deductive reasoning problems, most children recognized that the problems involved an arrangement of items or a matching of names with items. As Kerry, a low-achieving 9 year-old stated, "In the books' problem, you had to stack them and in the cards' problem you had to arrange them across." Most children were also able to recognize the similarity in item arrangements, for example, "The houses problem is like the cards problem because you have to work out which ones go next to each other. And the tower (of blocks) is like this one (stack of books) because you have to stack them up in the right order" (Hayley, 9 year-old low achiever).

Few children however, commented on the nature of the clues per se, such as the extent of information they provided, or the need to look for related clues. James, a high-achieving 12 year-old commented on the fact that there was one clue which provided a starting point: "The five houses along the street is like the cards problem because you knew where one was and then you had to figure out where the others would go.... there's sort of a trick to it. You got one of them (referring this time, to the stacking problems) and you had to figure out which went on top and which went below." It is worth mentioning the response of 12 year-old Natalie when asked if solving one set of deductive reasoning problems helped her solve the other. She claimed, "I did each (set of problems) separately. I didn't relate them." When questioned on the similarities between the problem sets, she commented, "You've got to match stuff up with other stuff but otherwise I don't relate problems as I don't really look at that sort of thing."

Many studies have shown that novices tend not to focus on the structural features of isomorphic problems especially when they have different surface features or when the surface details provide misleading cues (e.g., Novick, 1992; Reed, 1987). This highlights the importance of clarifying the source structure for children and ensuring they recognize the similar relations between the source and target examples. This applies to all areas of analogical reasoning, as the following principles indicate.

PRINCIPLES OF LEARNING BY ANALOGY

The following principles draw upon some of Gentner's (1982) criteria for effective analogs.

Clarity of Source Principle

The structure of the source should be clearly displayed and explicitly understood by the child.

For an analogy to be effective, children need to know and understand the objects and relations in the base. It is particularly important that the child abstracts the structural properties of the base, not its superficial surface details. It will not be possible to map the base into the target, then use the base to generate inferences about the target, unless this understanding has been acquired and is readily available.

Clarity of Mappings Principle

There should be an absence of ambiguity in the mappings from base to target.

The child should be able to clearly recognize the correspondence between base and target. When a base has to be recalled from memory, it should be retrieved in terms of its generalizable structure rather than in terms of particular surface details (Gholson, Morgan, Dattel, & Pierce, 1990). This is particularly important in the development of abstractions. These are formed from mappings in which the source, itself, is an abstract relational structure, with few or no attributes. Hence if children are to form meaningful abstractions, they must learn the structure of the examples they experience. Good analogs can assist here because mapping between an analog and a target example encourages children to focus on the corresponding relations in the two structures (Halford, 1993).

Principle of Conceptual Coherence

The relations that are mapped from source to target should form a cohesive conceptual structure, that is, a higher order structure.

According to Gentner's (1983) systematicity principle, relations are mapped selectively, that is, only those that enter into a higher order structure are mapped. For example, in using various concrete analogs to illustrate grouping by ten, attention must be focussed on the corresponding relations between the groups of items, not between the materials themselves (e.g., the physical size relation between a bundling stick and an MAB mini is not mapped).

Principle of Scope

An analogy should be applicable to a range of instances.

Analogies with high scope can help children form meaningful connections between mathematical situations. For example, the "sharing" analogy in teaching the division concept can be applied readily to both whole numbers and fractions. Likewise, the area model can effectively demonstrate a range of fraction concepts and procedures.

These principles prove to be particularly useful in assessing the effectiveness of the analogs (both concrete and mental) used in children's mathematical learning. The analogs serve as the source while the concept to be acquired is the target. The value of these analogs is that they mirror the structure of the concept and thus enable the child to use the structure of the analog representation to construct a mental model of the concept. Due to space limitations, we present an analysis of only a couple of these analogs. Analyses of other analogs can be found in English and Halford (forthcoming) and English (submitted).

CONCRETE ANALOGS

Colored Counters or Chips

Colored counters and other simple environmental items, used in the study of number and computation, do not possess inherent structure as such, that is, they do not display in-built numerical relationships. However they can effectively demonstrate the cardinality of the single-digit numbers. In this instance there is just one mapping from the base (the set of counters) to the target (the number name). When applied to the learning of basic number concepts, colored counters score highly on clarity of source structure and mappings. When used with the appropriate language and manipulative procedures, these analogs can promote a cohesive understanding of single-digit numbers and of the elementary number operations.

The complexity of this analog increases significantly however, when it is applied to the development of place-value ideas. In this instance the analog takes on an arbitrary structure in order to mirror the structure of the target and, as such, the mappings between the source and target become more complicated. This implied structure is of a grouping nature where groups of counters or chips of one color are traded for a chip of a different color to represent a new group. This single chip represents a number of objects rather than a single object. The analog thus becomes an abstract representation because the value of a chip is determined only by its color, which is arbitrary, and not by its size. Because there is no obvious indication of each chip's value, there is not a clear mapping from the base material to its corresponding target numeral. In fact, there is a two-stage mapping process involved, namely, from chip to color, then from color to value. That is, the child must firstly identify the color of the chip and then remember the value that has been assigned to that color (the same situation exists with the Cuisenaire rods). This naturally places an additional processing load on the child, especially if she does not readily recall this value. Given the lack of clarity in its source structure and the multiple mappings required, this material does not seem an appropriate analog for introducing grouping and place-value ideas. It appears more suitable for enrichment work.

The counters analog also increases in complexity when it is used as a source for the part/whole notion of a fraction. For example, to interpret the fraction of red counters in a set comprising 3 red and 5 blue counters, the child must initially conceive of the set as a whole entity to determine the name of the fraction being considered. An added difficulty here is that the items do not have to be the same size or shape (in contrast to an area model comprising, say, a rectangle partitioned into 8 equal parts). Hence the child must see the items of the set as equal parts of a whole, even if the items themselves are unequal. While keeping the whole set in mind, the child must identify all the red counters and conceive of them as a fraction of this whole set. Since it is difficult to ascertain the whole and the parts, which more or less requires simultaneous mapping processes, it is not uncommon for children to treat the red and blue counters as discrete entities and interpret the fraction as a ratio (i.e., "3 parts to 5 parts;" Behr, Wachsmuth, & Post, 1988). It is for this reason that the analog comprising sets of counters is inappropriate for introducing the part/whole construct (Hope & Owens, 1987).

In sum then, colored counters do have considerable scope and can be an effective analog for early number and computation activities where there is clarity of source structure and unambiguous mappings between source and target. When the target concept increases in complexity however, the analog also becomes more complex and does not mirror the target as readily as before. The analog adopts an implied structure which makes it difficult to form clear and unambiguous mappings between source and target. In the case of the fraction example, the analog's structure encourages children to focus on the inappropriate relation, namely, the relation between the two colored sets instead of the relation between one colored set and the whole set. This means the analog does not establish the conceptual coherence required. However when used in conjunction with other fraction analogs (such as area models) and when accompanied by the appropriate language and manipulative procedures, this particular analog can enrich children's conceptual understanding of the fraction concept.

Base-ten Blocks

The base-ten blocks are probably the most commonly used analogs in the teaching of numeration and computation. Because the size relations between the bocks clearly reflect the magnitude relations between the quantities being represented, the blocks display clarity of source structure and clear mappings to the target concept. The analog also demonstrates high scope since it can be applied to a range of instances. For example, when used in conjunction with a place-value chart, the base-ten blocks can assist children in their understanding of the numeration of multidigit numbers. The blocks can also demonstrate the regrouping and renaming of whole numbers, and hence, can foster conceptual coherence of our numeration system.

While the blocks represent a highly appropriate analog, their effectiveness will be limited if children do not form the correct mappings between the analog representations and the target concepts and between their manipulations with the analog and the target procedures. The nature of the teacher's and children's explanations during the learning sequence is a crucial component here (Fuson, 1992; Stigler & Baranes, 1988).

While the base-ten blocks serve as an effective analog for whole numbers, they take on an added complexity when representing decimal fractions. Changing the values of the blocks to accommodate decimal fractions poses a higher processing load for the child. For whole numbers, the values assigned to the blocks normally remain fixed and children associate a given block with its whole number value. When the blocks take on new values, children are faced with additional mapping processes. For example, if the flat block is assigned the value, one unit (or one one), the long block is equal to one tenth and the mini, one hundredth. This means that, to interpret the representation for the number, 1.11, children must firstly identify the flat block as representing one whole unit. They must then recall that the flat block is equivalent to ten long blocks as well as one hundred mini blocks. Next, children have to perceive the long block as equivalent to one tenth and the mini, one hundredth, of the flat block. This process, itself, involves an application of the fraction concept. Once the respective values of the blocks have been established, children must interpret the decimal fraction being represented. If children do not make all of the mappings required, there is the danger that they will interpret the decimal fraction as a whole number, record it as such, and simply insert a decimal point.

The complexity of the mapping processes involved here means that the base-ten blocks can lose clarity of both source structure and mappings when used as an analog for the initial representation of decimal fractions. Since children have to apply an understanding of fractions in interpreting this analog, it would seem more appropriate to employ less complex analogs, such as partitioned region models, in introducing decimal fractions and reserve the base-ten blocks for application activities.

We now turn to a brief consideration of more abstract analogs, namely, established mental models, which serve as the source for the learning of a new target concept or procedure.

MENTAL MODELS AS ANALOGS

The term, mental models, has been interpreted variously in the psychological literature (e.g., Johnson-Laird, 1983; Halford, 1993; Rouse & Morris, 1986). As used here, mental models are cognitive representations which are active while solving a particular problem and provide the workspace for inference and mental operations (Halford, 1993). Because mental models comprise representations, and since analogies are mappings from one representation to another, mental models can serve as analogies. In using mental models as analogs, children need to explicitly recognize the correspondence between their model of a particular mathematical construct (i.e., the source) and the targeted construct.

Consider for example, children's learning of the relationships inherent in our place-value system. Children's introduction to multidigit numbers presents a new relational construct for the child, namely, the periods within our number system. The important feature of these is that the same set of relationships exists in each period, that is, the ones' period comprises hundreds, tens, and ones of ones, the thousands' period comprises hundreds, tens, and ones of thousands, and the millions' period, hundreds, tens, and ones of millions. This is readily demonstrated on a place-value chart. A meaningful mental model of the "hundreds, tens, and ones" relations within the ones' period can serve as an effective analog for the learning of larger numbers. This involves a process of mapping the "hundreds, tens, ones" model onto each new period within the number and assigning the appropriate period name. This is a less complex process for the child than the common procedure for reading numerals identified by Fuson, Fraivillig, and Burghardt (1992), namely, a reverse right-to-left process in which the child looks along the digits from right to left in order to decide the name of the farthest left place and then reads the number name from left to right. In sum, a mental model of the "hundreds, tens, ones" relations within the ones' period can serve as an effective analog for the learning of multidigit numbers since it displays clarity of mappings, that is, it is readily mapped onto each period of a multidigit number. The analog also promotes conceptual coherence of our number system because it highlights the important place-value relations.

CONCLUDING POINTS

This paper has examined analogy as a general model of reasoning and has proposed a number of principles for learning by analogy. The role of analogy in children's novel problem solving and in their basic mathematical learning was addressed. An analogy was defined as a mapping from one structure, which is usually already known (the base or source), to another structure that is to be inferred or discovered (the target). Mathematical analogs range from elementary concrete models such as counters, to abstract mental models such as place-value relations and algebraic relations. The value of these analogs is that they mirror the structure of the targeted mathematical idea and thus enable children to use the structure of the analog representation to construct a mental model of the new idea.

The important feature of analogies is that the structural correspondence between the source and target is mapped, not the superficial attributes of these elements. Relations are mapped selectively, that is, only those relations that enter into a coherent structure are mapped. One of the values of analogies is that they transcend domains which may be very different, apart from the relations they have in common. Since analogies focus on common relational structures, reasoning by analogy is an important process in children's mathematical learning. On the other hand, analogs can often possess inherent or arbitrary structures which can detract from their effectiveness. Effective analogs are those in which the child clearly recognizes and understands the structure of the source, can clearly recognize the correspondence between source and target, and can make the required mappings from source to target. Analogs which are applicable to a range of instances can help children form meaningful connections between mathematical ideas.

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