PERSONAL RECONSTRUCTION OF CONCEPT DEFINITIONS: LIMITS

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Students' difficulties in learning about limits have been documented by many authors. Cornu, Tall, and Dubinsky all emphasise the process of encapsulation, whereby a dynamic process becomes transformed into a static concept image. Tall and Gray also place importance on the "proceptual" nature of mathematical thought. I report on the early stages of a study that is designed to access student schemas in limit problems.

25 students in a second year university numerical analysis class were given a questionnaire that probed aspects of their understanding of limits. The schemas of 5 of these students will be examined using clinical interviewing techniques over a 4 week period. The numerical analysis course uses spreadsheets instead of a programming language, and one of the longer-term aims of this research is to determine the effect of regular use of a spreadsheet on students' concept image of limits.

THE TEACHING EXPERIMENT

Problems with learning about limits have been documented recently by a number of authors - Cornu (1991), Dubinsky and Tall (1991), for example. These authors, in particular, stress the importance of the transformation of a mathematical process into an object. The research reported in this article consists of the first 4 weeks of a study designed to access the action schemes of tertiary students in a computer-based numerical analysis course, with emphasis on their notions of limits.

A questionnaire addressing students concepts of limits was given to an entire class of 25 students in a second year university numerical analysis course. Five of these students volunteered to take part in a longitudinal teaching experiment. The teaching experiment is carried out once per week for each student. Each session lasts approximately 40 minutes. A worksheet designed to access and build on demonstrated action schema is presented at each session. Problems are structured to allow implementation on a computer if required. The students are given 20 minutes in which to solve these problems, followed by a group discussion of the problems solved and schemes used. All sessions are video-taped for later analysis.

The methodology of the experimental work is based in constructivist theory and in the use of a constructivist teaching experiment in particular (Opper, 1977; Cobb and Steffe, 1983; Hunting, 1983; Steffe, 1984; Davis and Hunting, 1991). This methodology emphasises detailed observation and documentation of action schemes and their re-organization in problematic situations. Dubinsky (1991) says:

> "It is not possible to observe directly any of a subject's schemas or their objects and processes. We can only infer them from our observations of individuals who may or may not bring them to bear on problems - situations in which the subject is seeking a solution or trying to understand a phenomenon. But these very acts of recognising and solving problems, of asking new questions and creating new problems are the means by which a subject constructs new mathematical knowledge." (p. 103)

SPREADSHEET ENVIRONMENTS AND INTERNALIZATION OF SCHEMES

Spreadsheets

The numerical analysis course is based on using spreadsheets rather than on a programming language. This is to reduce the amount of time spent debugging programs and to allow students to rapidly engage with the numerical analysis aspect of the course. A spreadsheet provides an action-oriented environment in which a teacher/researcher should be able to observe and document the creation of mathematics concepts by students. Dubinsky (1991) claims:

> "... if a student implements a process on a computer, using software that does introduce programming distractions, then the student will, as a result of work with computers, tend to interiorize the process. If that same process, once implemented, can be treated on the computer as an object on which operations can be performed, then the student is likely to encapsulate the process." (p. 123)

Interiorization and internalization

Steffe's (1988, p. 337) use of the word "interiorization" is apparently different to Dubinsky's use of it in this context. Dubinsky's "interiorization" seems to be closer to Steffe's internalization. The difference, for Steffe, is this:

• interiorization is a very general form of abstraction that leads to the isolation of form, coordination, and actions from experiential things and activities; an interiorized entity is devoid of sensory-material;

• internalization: a process that results in an ability to re-present (as in a video-like image) a sensory item without the sensory signals being present.

From this perspective it seems that Dubinsky is talking about the weaker term "internalization". There is no evidence from his statement that a student who has encapsulated a process has abstracted a general pattern or structure sufficient to use it without visual representation.

STUDENT SCHEMES

The focus of this study is on describing and analysing students' schemes when they work with limit problems in a computer-based environment. There were 4 schemes that were used extensively by the students in the study. They were:

SCHEME:	TYPICALLY INVOLVES:
algebraic	manipulation of rational expressions
numerical	iterative calculation of values
neglecting small terms	erasing terms that approach 0
dominant terms	focussing on dominant terms and neglecting all others

What is striking about their algebraic and numerical schemes in particular is the lack of *co-ordination* of these schemes in individual students minds.

Algebraic schemes

Two students - Jane and Michael - were given the following problem:

> Use of a spreadsheet may be made if required.
>
> $s(n) = s(n-1) + 2s(n-2)$
>
> $s(0) = 1$
>
> $s(1) = 1$
>
> Does $\frac{s(n)}{s(n-1)}$ have a limit as n approaches infinity?

They were sitting at a desk, with another student Simon, where there was a computer turned on with a spreadsheet, not showing but available for them to use (the same situation that holds in their lab work). Jane and Michael approached the problem by algebraically manipulating the recurrence relation.

Jane's calculations:

$$\frac{s(n-1) + 2s(n-2)}{s(n-1)} = 1 + \frac{2s(n-2)}{s(n-1)}$$

$s(n-1) > s(n-2)$ for all n.

$$\lim_{n \to \infty} \frac{s(n)}{s(n-1)} = 1$$

Jane's algebraic method is one she has used in previous problems. She adopts the method of ingnoring terms if they become insignificant after a while: neglecting all terms that go to 0 without taking into account the rate at which they go to 0. This scheme has worked fine for her in previous problems but is inappropriate and inadequate for this problem.

Michael's calculations:

No

$$\frac{s(n)}{s(n-1)} = 1 + \frac{2s(n-2)}{s(n-1)}$$

$$= 1 + \frac{2(s(n-3) + 2s(n-4))}{s(n-1)}$$

$$= 1 + \frac{2s(n-3)}{s(n-1)} + \frac{4s(n-4)}{s(n-1)}$$

$$\frac{s(n)}{s(n-1)} = 1 + \sum_{x=1}^{n-2} \frac{2^x s(n-2-x)}{s(n-1)}$$

Michael then stated that the 2^x term will increase exponentially at a much faster rate than the remaining part of the sum, so there will not be a limit.

For them, it seemed more important being able to calculate algebraically rather than implementing the problem on a spreadsheet. We could argue here that by the 4th week of the course, the action schemes required by the spreadsheet environment are only in the process of become internalized. However, Michael has an honours degree in engineering, he is at the top of his group in the numerical analysis course, and he was the top student in first year applied mathematics. He has used spreadsheets extensively in his engineering course and for relatively deep problems - the simulation of feedback systems using discrete time-input signals, for example. Jane is an outstanding undergraduate who came top in a third year mathematics component in her first year. At the end of first year she gained a Mathematics Department vacation scholarship to study new ways of calculating Hausdorff dimensions of fractal sets. Jane used spreadsheets in a year 12 physics experiment and a statistics project. So the evidence is that these capable students have had reasonable prior experience with spreadsheets.

What seems more likely is that their algebraic schemes are very dominant: these schemes have been extremely powerful for them in the past. I conjecture that these two students algebraic schemes have passed beyond the internalized stage and have become interiorized.

Numerical schemes

The next question for Jane and Michael was:

> How does the ratio $\frac{s(n)}{s(n-1)}$ vary with n for n = 0 to 10?
>
> Use a spreadsheet if required, but only compute the ratio
>
> $\frac{s(n)}{s(n-1)}$ for n = 0, ..., 10.

They approached this problem by filling down the recurrence on a spreadsheet. They stated from the numerical evidence that the ratio $\frac{s(n)}{s(n-1)}$ approaches 2, but they couldn't say why. Michael could see this, but was reluctant to accept it in the light of his algebraic calculations. He appeared to be in a state of conflict. In a later problem Michael came up with a conjecture and used the computer to check it. This is evidence that he is prepared to use numerical evidence in support of a conjecture, but has considerable difficulty when the evidence seems to refute his conjecture.

Why didn't Michael have faith in the spreadsheet calculations? Why didn't he just accept the evidence? This phenomenon is widespread in mathematics: a reluctance to abandon a schema in light of evidence that is inappropriate. This is not surprising, for if Michael is to abandon his algebraic scheme, with what is he to replace it? He is in a state of disequilibrium in which his world-view is no longer stable.

Disassociated schema

I feel the students have internalized algebraic methods of solving for limits, which is taught in first year without reference to a concept definition of a limit. The algebra of limits has become the paradigm in which they work. This internalization, which allows students to solve problems more automatically, then becomes their habitual and preferred way of acting. Their development of a deeper and more structured concept of limit is deferred. This is, of course, fine for them until their algebraic scheme becomes inappropriate and entangles them, as it did in he first problem they were given. Dubinsky and Tall (1991) say:

> " ... many researchers have realised that student errors are often the product of misconceptions brought about using old knowledge in a new context where it no longer holds good." (p. 234)

Jane and Michael's algebraic schemes are not "misconceptions" in the usual sense of the word: rather they are *inadequate* conceptions. And they are inadequate only in that they lead to massive entanglement which prevents them from coming to a tenable conclusion. Both students came to a conclusion as to the limit - Jane stating it was 1, and Michael stating there was no limit - but their conclusions were not tenable in light of the numerical evidence.

There is no doubt that an integration of numerical and algebraic schema *can* lead to very powerful and rapid conclusions. For example, once the limit of the ratio $\frac{s(n)}{s(n-1)}$ is recognised as 2 from the numerical data, an algebraic calculation shows that $\frac{s(n)}{s(n-1)} - 2 = \frac{s(n) - 2s(n-1)}{s(n-1)}$. A numerical experiment shows that $s(n) - 2s(n-1)$ is ± 1. This is easily checked algebraically, so the numerical and algebraic schemas associate to lead to a conclusion that $\frac{s(n)}{s(n-1)}$ approaches 2 by oscillating about 2 with an error equal to $\frac{1}{s(n-1)}$. This very simple association of schemes is incredibly powerful.

None of the 5 students in the study demonstrated any actions related to determining the *error* between the terms of a sequence and its purported limit. The only time they mentioned the error was when they observed oscillatory behaviour around the limit from the spreadsheet data. One of the students, Lindsay, said, when asked why he had not checked the sequence of error terms, that he did not know what the limit was, despite the numerical evidence from the spreadsheet in front of him. In the numerical analysis classes students are given a sequence and its limit in advance in order to analyse the error terms: the students are not told that the number given as the limit is, in fact, the limit, but are just asked to calculate the error terms.

CONCLUSION

The schemes that we saw in operation with these students were the following:

- Algebraic schemes, which typically involve manipulation of rational expressions.
- Numerical schemes, which typically involve iterative calculation of values.
- A "neglecting small terms" scheme in which terms that approach zero are erased.
- A "dominant terms" scheme in which students focus on dominant terms and neglect all others.

These are all powerful schemes. However they all also have limitations, and for the students in this study, the biggest hurdle to their effective use is the *integration* of these schemes, one with another. I have presented evidence that students often use schemes in a disassociated way.

The two students reported on in this study operate as if algebraic calculations produced "real" or firm evidence, whilst numerical calculations only provide partial or tentative evidence. A problem for them is to see how to flexibly integrate these schemes to relate the algebraic and numerical data: their algebraic manipulations lead them to false conclusions or to no conclusion at all as to a limit, whilst their numerical spreadsheet calculations show them a limit, but give them no reason to believe why it is the limit.

The students' algebraic schemes seem to very stable and very well-integrated; it may well be, although there is no evidence of this yet, that their algebraic schemes have become genuinely interiorized, whereas there is evidence that this is not true of their numerical schemes.

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