TRANSFER OF ABSTRACT THINKING IN MATHEMATICS

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Abstract:

Mathematics teachers' influence on student learning of mathematics could interfere and limit the learning of higher order mathematics. This paper stages this influence as a concern for teacher educators. To assess the affects of this influence, 1st year University mathematics students were selected as a study sample. These students mathematical understanding was explored using 'mathematical items' designed specifically for this study. The students' responses were assessed and evaluated using the SOLO taxonomy. This paper also reports on preliminary findings focusing on the transfer of abstract thinking in functions. The findings tentatively suggest that prior learning affects the depth and clarity of University students' understanding of mathematics.

Mathematics by its nature is generally a subject of learning that progresses sequentially in a hierarchical fashion with high levels considered to be more abstract than previous ones. Secondary mathematics teachers are assumed to have achieved these high levels of abstract learning (thinking) and to possess the ability to transfer these abstract learning to the learners. However, according to the recent nationwide discipline review of Teacher Education in Mathematics and Science (DEET,1989), one of the areas requiring attention is the 'ability of secondary mathematics teachers to transfer abstract thinking' (p13).

Teacher educators have a responsibility to define what constitutes abstract thinking and knowing how and when transfer has occurred. In this study, **transfer** is defined as one's ability or potential to explicitly communicate substantiative knowledge of subject matter, in this case mathematics. **Abstract thinking** is defined as high order knowledge and understanding about the subject matter.

Context:

The literature portrays a growing concern of our children's lack of appreciation (or poor attitude) and poor understanding of mathematics. To address this concern there is a need for teachers who are confident in their own mathematical knowledge and who themselves have a grasp of mathematical concepts and ideas. According to Even (1993) the process of learning is influenced by the teacher and the teachers' pedagogical content knowledge is influenced by their subject matter knowledge. Leder (1991) concluded that teachers' poor grounding in mathematics could be blamed on students' difficulties in understanding mathematics. Further, she suggested that a shortage of well qualified mathematics teachers at all levels of the educational system continues to be a matter of concern.

The training and education of secondary/primary mathematics teachers is an important link in the mathematics education of our children and the area of interest for this research. However, the negative experience with mathematics that many students have during their pre-tertiary schooling is having serious effect when they come to consider studying mathematics in teacher education courses. According to McAuliffe (cited in Leder,1991), the collapse of student enrolment in mathematics education is a real crisis in teacher education. The downfall in student enrolment in mathematics education is a real concern because it means that any improvement in the quality of mathematics teachers' education can only be transmitted into the system by the very few who, given the current state of teacher employment, may not themselves have the opportunity to practice. After all, it is how the teacher is prepared for the tasks of education which will, to a tremendous degree, affect student outcomes (Porter, 1989).

The Schools Council of the National Board of Employment, Education & Training (NBEET,1989,p16) stated that " the quality of education is inescapably related to the quality of teachers ... including the academic quality of students entering pre-service courses of teacher education". However, tertiary institutions seem to have other objectives when it comes to admitting students into secondary mathematics education courses. For example, a study conducted by the Ministry of Education in W.A. (1992) recorded that 44.1% of students enrolled in Secondary Mathematics Education program in tertiary level have failed pre-tertiary mathematics units or not undertaken any such units. Such practices are contrary to the notion that knowledge of subject matter for teacher training and education is important (Ball,1990). Knowledge of subject matter, in this case of mathematics entails substantiative knowledge of mathematical concepts and procedures, understanding the underlying principles and meanings, and an appreciation and understanding of the connections among mathematical ideas.

Approach

The research began with the need to identify the levels of mathematical learning and understanding that the pre-service secondary mathematics teachers have been, in the past, assumed to have acquired during their pre-tertiary schooling and education. Piaget's theory of 'cognitive development' was used to examine and assess the levels of mathematical understanding of the pre-service mathematics teachers. However, according to Biggs & Collis' (1982) SOLO taxonomy, there are levels of abstract thinking in each developmental stage (referred to here as Piagetian stages). The SOLO taxonomy was therefore, selected as the theoretical basis for this study. It also provided the basis for an instrument to evaluate and assess the 'responses' to study items designed specifically to explore the transfer of abstract thinking in mathematics.

The five key levels of the SOLO taxonomy are: Pre-structural, Uni-structural, Multi-structural, Relational and Extended Abstract. These five levels are said to occur to some degree in each of Piaget's developmental stages. According to Collis & Biggs (1983), entry into university requires students to demonstrate the ability to respond in the formal (1st order) mode (Piaget's top developmental stage), and at least the uni-structural level (lower levels of the SOLO taxonomy).

To explore the transfer of abstract thinking in mathematics, and specifically at the tertiary/university level, suitable 'mathematical items' were required. They were developed and piloted with a sample of first year University mathematics students, experienced mathematics teachers and mathematics lecturers. The initial responses were then assessed and examined using the SOLO taxonomy. Assistance was provided by one of SOLO's authors, Professor Collis, whose suggestions were adopted in the design of the final four 'items'.

The approach to item design adopted for this study was one that not only meets the conditions of the SOLO taxonomy (theoretical basis) but also allowed the respondents to express their understanding of the mathematical situation as they see it. Another key aspect behind the design of the items is the notion based on the recommendation that secondary mathematics teachers should relate mathematics to real world situations (DEET, 1989). Only one of the four items (Functions: Logarithm & Trigonometry, Negative Number, and Statistics) will be considered here, that is Logarithm.

Logarithmic Functions:

The influence of learning by teachers has been investigated by Even (1993) using functions. Even (1993) reported that the majority of the participants (student Maths teachers) define function as an equation, an algebraic expression, or a formula. Such definitions for functions are an indication of lower levels of abstract thinking for functions. Researchers (Crawford, Gordon, Nicholas, and Prosser, 1993) have reported that 77% of their sample of 1st year University mathematics students have conceptions of mathematics as numbers, rules and formulae with applications to problems. They also reported that 76% of the same sample's approach to learning mathematics was learning by doing examples (drill & practice). These results appear to agree with Collis & Biggs' (1983) suggestion that entry into University requires that students demonstrate the ability to respond in the formal (1st order) mode, and at least the uni-structural level (lower level). Assuming that the learning of mathematics is a 'hierarchical process', this study places knowledge and understanding of log functions at the top echelon of this hierarchy.

Logarithmic functions can be seen as 'mathematical models' for many real life situations. For example, logs can be used to simplify and determine models for research data on growth & decay (exponential occurrences) which have been conducted without prior 'models'. Often these exponential occurrences are easier to understand if they can be transformed or be modelled by a linear relationship made possible by the log function. For example: Let Y be the Exponential function:

$Y = ax^n$ or $\log Y = n\log x + \log a$ (linear transformation)

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An individual acquiring the knowledge of log function with the ability to clearly express (or show understanding of) its unique properties would suggest that such an individual has progressed satisfactorily through the preliminary learnings (ie. algebraic expression, formulae, etc.) of functions. In addition, such an individual would have potential to relate mathematics to real world situations.

There is, however, an expectation of teachers to know more than their students. It is important then that pre-service mathematics teachers show evidence of having higher levels of abstract thinking during their training and education. Attainment of these higher levels for functions would be evident when the respondent expresses properties unique to each function (eg. log, trig) and differentiates them from 'algebraic expressions'.

The 'log item' for this study strongly reflects the definition of functions as an 'algebraic expression' (see Appendix). An individual who has a sound understanding of algebra can actually feel the success of finding the solution, for example, to the following equation: **log(2X+10) = log X, solve for X.** In considering **log** as a <u>common factor</u>, a possible solution (algebraic method) is as follows:

$\log(2X+10) = \log X$

$\log 2X + \log 10 = \log X$ (expanding LHS)

$\log 2X - \log X = -\log 10$ (collecting like terms)

$\log X = -\log 10$ ($\log 2X = 2 \log X$)

$X = -10$ (dividing both sides by log)

Compare the above solution to another solution (algebraic method):

$\log(2X+10) = \log X$

$2X + 10 = X$ (recognising that $(2X+10)=X$)

$2X - X = -10$ (collecting like terms)

$X = -10$

This is often the algorithm taught to solve this type of **log** problem.

Both approaches have given the correct value of X=-10. The first solution is 'algebraically correct' but the process is a violation of log functions. Suppose both respondents discarded their 'working out' of the problem and only produced their final answers as X=-10, which one has the 'higher level' of understanding of log? The discriminator is knowing the unique properties of log functions. That is, although the value of X=-10 is 'algebraically correct', for a log function it has no meaning because the log of a negative number does not exist (in the field of Real numbers). However, the learner (individual) that does not have this knowledge and understanding (abstract thinking) of log functions would happily accept X=-10 or log (-10) as a valid solution.

It is suggested from what has been mentioned that the understanding of log functions requires more than a sound knowledge of algebra (lower abstract levels). There is also the suggestion that an individual cannot attain the 'top' successfully without a sound grasp of the 'lower levels'.

Study Trial Sample:

The trial sample were 54 first year University mathematics students (23 enrolled in Dip.Eng. & 31 B.Eng.) and three experienced mathematics teachers. The University students group was selected because of the assumption that these students are the successful and high achievers of mathematics during their pre-tertiary education. Also a subgroup of this main group will form the group of those who will be training to be mathematics teachers. The responses from this group will provide a good indication of the type of teaching and hence the type and level of 'abstract thinking' they have acquired. The sample's responses should also provide a good indication of how abstract thinking is being transferred via teaching from teacher to student.

Samples of Responses - categorised using SOLO:

1.A Pre-structural Response: To Q1, *No.* To Q2: *Because line 3: log 2X =logX , 2X = X, 2X - X = 0, X = 0 . Obviously replacing X=1 into the original equation the statement isn't true.*

This response is pre-structural because the respondent has been distracted and mislead by 'irrelevant' data. Since he/she can't find or make the right association of what he/she knows compared to the given, he/she finds the part of the given he/she can relate to, selects this and attempt to justify that what he/she knows and understands is true.

2.A Uni-structural Response: To Q1, *No.* To Q2: *In the 2nd line it should be: log(2X + 1) - log(X - 1) = 0. Keeping the unknown and the constant together then apply log laws, ie. logX - logY = log (X/Y).*

This response is uni-structural because the respondent has focused on the relevant domain, selects one aspect (eg. log(a+b) $\neq$ loga + logb) and the rest of the responses are justifications of this single aspect. The respondent can retrieve the correct knowledge about the given but can't proceed any further.

3.A Multi-structural Response:To Q1,_No._**To Q2:**_log(2X+1) = log(X-1), 2X + 1 = X-1 (as they have the same base), X= -2 . In the 'response' it looks like the student is looking for a problem that is too hard & not really there._

This response is multi-structural because the respondent has selected more than one relevant feature, but has not integrated them with the other cues. He/she has focused on the main situation (original equation) and ignored the task at hand (a student with understanding difficulties).

4.A Relational Response:To Q1,_No._ **To Q2**_$log_{10}[2(1)+1] \neq log_{10}(1-1)$ $log_{10}(3) \neq log_{10}(0)$. The student should not opened the brackets, it should be: $log_{10}(2X+1) - log_{10}(X-1) = 0$, $log_{10}(2X+1)/(X-1) = 0$, $(2X+1)/(X-1) = 10^0$, $2X+1 = X-1$, $X= -2$._

This response is relational because the respondent has integrated the cues with each other to form a coherent explanation. He/she has considered the situation as a whole. That is, selects the given 'wrong cues' and relates them to his/her own knowledge and understanding (referred to in this study as implied cues) and justifying each action so that the final result is not just an outcome of knowing (eg. log laws) but also an understanding of the processes required to give a solution.

Early data from this project, suggests that it is at this level (relational) where transfer of abstract thinking from teacher to student begins to take place or emerge. This transfer is recognisable by the occurrence of the following five steps:

Step1: The person considers the situation as a whole.

Step2: Analyse the validity/viability of the situation(eg.subsitution using the given information: for X=1, $log[2(1)+1] \neq log(1-1)$).

Step3: Focus on the situation or task at hand and determine what processes are involved.

Step4: Interpret the processes by relating it to prior learning and understanding.

Step5: Justify these processes (abstract thinking) by giving clear and logical, as well as true, responses or answers (eg. X=-2)

5.An Extended Abstract Response: To Q1,_No._ **To Q2:**_Going from step1 to step2 is wrong with logs as you can't expand. Step3 to 4 is correct, (ie. $log_{10} X = 0$ ...) . The actual answer is X= -2 but all logs should be checked in the original equation as they can't be negative, it is invalid._

This response is Extended Abstract because the respondent has taken the extra step (**6th step**) which clearly distinguishes it from the Relational level. This step is the recognition and identification of the properties which make the given situation (eg. log functions) unique or different to other situations in the same field (mathematics). That is, giving the correct solution (eg X=-2) as in Relational level or in the

Multi-structural level does not necessarily make the transfer of abstract thinking in logarithm complete.

To show that one has the **ability** for possible **transfer** and **potential** of attaining higher order abstract thinking, the respondent must display or come forth with the understanding of the **uniqueness of the situation** at hand (eg. it is invalid to have a log of a negative number).

Preliminary Results:

Classifying the 'responses' into SOLO levels was done using mapping procedures. The table in Figure1 shows a relative frequency distribution of the responses from the University student sample only to both function items (log & trig). The sample was split into their respective groups identified here as A,B,C. The A group does one unit of mathematics which is similar to B and C but without the rigour and depth. The mathematical background of those in A are considerably lower than those in B and C.

Figure 1: Relative Frequency Distribution: Logarithmic & Trigonometric Functions:

		SOLO LEVELS (FORMAL MODE 1)					
	GROUP	CON	PRE	UNI	MULT	REL	ABS
Log	A	44	44	4	4	4	0
Trig	n = 23	44	36	8	4	8	0
Log	B	8	31	31	15	15	0
Trig	n = 13	15	24	31	15	5	0
Log	C	0	11	28	28	28	5
Trig	n = 18	0	5	45	45	0	5

(All values are in percentages)

Since the data is very small for some of the SOLO levels as shown by Figure 1, the first 3 levels (concrete, pre-structural, uni-structural) are collapsed into one and called University Entry Point (UEP) to coincide with 'entry level' as recommended by Collis & Biggs (1983). The last 3 levels are also collapsed into one called the Desired Stages for Transfer (DST). These figures are summarised in Figure 2.

Figure 2:

Log Function			Trig Function		
Grp	UEP	DST	Grp	UEP	DST
A	91%	9%	A	87%	13%
B	69%	31%	B	69%	31%
C	39%	61%	C	50%	50%

These figures tentatively suggest that prior mathematical knowledge has an influence in mathematical understanding of higher order levels. The large proportions of the A and B groups at the UEP stage is of interest to the mathematics teacher educators, because coming through these groups, particularly the B group, are the secondary mathematics pre-service teachers. These figures also seem to indicate that for this sample to reach their potential level of transfer they would need to further their mathematical knowledge and understanding, that is, in both depth and clarity.

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APPENDIX: LOGARITHMIC FUNCTION MATHEMATICAL ITEM

SIMPLIFY AND EVALUATE FOR X,

$$\log_{10}(2X + 1) = \log_{10}(X - 1)$$

A Student's response:

$$\log(2X + 1) = \log(X - 1)$$
$$\log 2X + \log 1 = \log X - \log 1 \qquad (\log 1 = 0)$$
$$X = 10^0 \qquad (\log_{10} X = 0)$$
$$X = 1$$

Q1. Is the student's response correct?

Q2. Please explain and show (space below) why you answered yes/no to Q1.