ABSTRACTION AS THE RECOGNITION OF DEEP SIMILARITIES: THE CASE OF THE ANGLE CONCEPT

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Thirty six Year 2 children were each presented with two out of a set of six realistic models of various physical angle contexts and asked (1) to indicate which of a set of 10 abstract angle models could also represent those contexts; (2) to represent the contexts in drawings; and (3) to indicate whether they recognised any similarities between the two contexts. Previous papers have reported results for the turns, slopes, rebounds and corners contexts. The present paper reports on the crossings and bends contexts and summarises the specific features of each of the 6 contexts which appear to hinder the recognition of similarities. The basic hypothesis of abstraction theory — that an abstract angle can only be recognised in a particular context when angle-related similarities between that context and other, superficially different contexts are also recognised — is then examined.

The research to be described below arises from a model of conceptual development in mathematics developed by Paul White and myself (White & Mitchelmore, 1992). This model is based on the assumption that children develop mathematical concepts by abstracting the common features of various situations and learning to ignore the specifics (Skemp, 1986). But concept formation is not a once and for all process; as more and more dissimilar situations are seen to contain the same common elements, the concept becomes more and more general (Mitchelmore, in press). A common path is for a concept to develop separately in different contexts which the learner does not link together because of superficial differences; generalisation occurs when the superficial differences are seen to be less important that the deep structure.

The basic hypothesis of abstraction theory, as I shall call it, is that **abstraction necessarily follows recognition of similarity.** For example, an abstract concept of angle cannot be meaningfully constructed from experience in a single context but requires the learner to coordinate angle experiences in several superficially different contexts.

Elsewhere, I have described 14 angle contexts which would seem *a priori* to be superficially different (Mitchelmore, 1993a). In an initial investigation, 6 were selected and presented to a sample of Year 2 students. An interview was designed to investigate how far children had abstracted an angle concept from each context, and whether they recognised the same concept in different contexts. Previous papers have reported on the rebound and corner contexts (Mitchelmore, 1993b) and the turns and slopes contexts (Mitchelmore, 1993c). The present paper reports on the last 2 contexts, crossings and bends, and on the similarities which students recognised between all 6 contexts.

METHOD

A sample of 36 children, average age 7.4 years, was selected from 2 Catholic schools in Sydney. Each child was interviewed on 2 of the 6 contexts; 9 pairs of contexts, each administered to 4 children, were chosen in such a way that a total of 12 children responded to each context.

The crossings context was presented in the form of a model pair of scissors made of 2 narrow strips of wood pivoted about a point one-third the way along each strip. After talking about scissors in general, children were asked to indicate the maximum and minimum openings and to say what determines how far they open scissors in normal practice.

The bends context employed a plan of an imaginary town drawn on a card 520 mm by 820 mm. A road 40 mm wide, consisting of 9 straight segments ranging in length from 110 mm to 360 mm, formed a closed loop around the town. The straight segments were joined by short circular curves (internal radius 30 mm); the 9 (interior) angles between successive straight segments ranged from 15° to 140°. Children were asked to drive a toy ambulance as fast as they could from a "factory" on one side to a "hospital" on the other, and to state which bends were the hardest and which were the easiest to drive around. They were then asked to say how they could tell whether a bend would be hard or easy, and to indicate the hardest and the easiest imaginable bends.

After these different initial introductions, intended to test children's concrete understanding of each context, the interview proceeded in the same way in both contexts. Firstly, in an attempt to assess the extent of each context presented, the interviewer asked children to name examples of anything else which "opens and shuts like scissors" or "bends like this road". Secondly, children were shown a set of ten abstract angle models made out of plastic or straws; models 1-3 showed a single line rotating on a fixed background, models 4-6 showed 2 lines rotating relative to each other, models 7-9 were made of pairs of straws joined by pipe cleaners; and model 10 showed a variable sector. In models 4 and 7 the 2 lines crossed, in models 5 and 8 the end point of one lay on the other, and in models 6 and 9 the 2 lines had a common end point. (For illustrations, see Mitchelmore, 1993b). The interviewer asked children which of these abstract models could be used to show scissors or bends, and to select the one they thought was the best model for doing this. Children were asked to demonstrate how each selected model showed scissors open a normal amount or a particular bend in the road, and how their best choice showed the extreme openings or bends. Thirdly, children were asked to draw various openings or bends.

Similar procedures for the other 4 contexts are detailed in Mitchelmore (1993b, 1993c).

After children had responded to their 2 models, the interviewer asked "Are these two the same, at all?" Children were then asked whether and how their best abstract model of each context could also show the other context; if they had chosen the same best model, this was simply pointed out to them. The interviewer then asked children to say what was the same about the 2 contexts.

body) were bends which could be modelled by 2 jointed line segments; 8 were flexible objects such as a plastic ruler which bend in a continuous manner; and 12 could bend in both ways.

Students had considerable difficult finding how to use the abstract models to represent bends in the road. Models 1-3 were selected 34% of the time, but very few explanations referred to an angle; 2 students referred to turning the line first along one road and then along the other, but most saw the circle as a steering wheel or used part of its circumference as the bend. Models 4-10 were selected 72% of the time, but only 16% of all responses matched the arms of the models to the 2 straight segments of the road at the bend. Only one student used all of models 4-10 correctly; no other student had more than 2 correct. The most popular best abstract model was number 10, chosen by 5 students; however, although 4 students could use their best model to show the easiest bend imaginable, no student could show the most difficult bend imaginable.

Students invented 2 main non-standard methods of representing bends using abstract models 4-10. One was to use the edge of the circles to model the curve (5 students); a further 2 students rejected models 7-9 because they could not bend the straws. The second strategy, which might be regarded as nearer to the standard angle representation, was to place the lines of the model somewhere on the road at the bend without attempting to align the lines to the roads (6 students). Some students set the vertex at a point on the outer curved edge of the road, some set the lines to touch the inner curved edge, and others were content to place it anywhere within the bends. Two of these students rejected at least one model because the lines were too long to fit into a sharp bend.

Eight out of the 12 children drew at least one bend as a continuous curve without any straight segments at the ends, and none draw a bend emphasising the 2 line segments. Only 2 children represented the roads abstractly by single curves. Where straight segments were present (20 drawings), the angle between them was measured. For drawings of the sharpest bend (interior angle 15°), the 2 line segments were in all cases parallel; 4 out of 7 drawings of a 60° bend were within 10°; and 3 out of 8 drawings of a 125° bend were within 10°, 2 being clearly acute angles.

In summary, it would appear from children's drawings, method of abstract modelling and selection of similar situations that they did not conceptualise a bend in the road as 2 line segments joined by a relatively insignificant curved part. The curved part of the bend was the most salient, and the straight parts tended to be overlooked. The word "bend" is indeed ambiguous, and a cursory glance at a city road map will confirm that although most roads do consist of lengthy straight segments connected by short curves, there are also many long curved stretches. (I considered using "corner" instead of "bend", but apart from confusion with tile corners, felt that this term was restricted to sharp bends.) In the circumstances, it is perhaps not surprising that students were so inaccurate in dealing with the sizes of the bends; it was also noted that most errors in identifying the hardest and easiest bends consisted of choosing the second hardest and the

RESULTS FOR CROSSINGS AND BENDS CONTEXTS

Crossings

The crossings context proved to be very easy for most students. All students explained that the opening of the scissors depended on the length of the line or the thickness of the object to be cut, and all correctly indicated the extreme open and closed positions of the model. All 12 children could name several examples of things like scissors. Of the 34 responses, all but 6 involved 2 lines crossing each other. However, only 6 of these examples actually opened and closed in practice; 9 examples were static crossings which could take on various shapes (6 students suggested a letter X) whereas the remaining 13 examples were all right angles (9 students mentioned a cross).

All of the children rejected abstract models 1-3, saying, for example, that they only showed one arm of the scissors. Half of the children accepted all the remaining models 4-10. The other 6 all rejected model 5 on the basis that "it doesn't have a handle" but only 3 of these also rejected the geometrically isomorphic model 8. Two children expressed reservations about some models because the arms were not quite the same length. All the children's best models appropriately modelled the various scissors angles using 2 intersecting lines; 6 children chose model 7, 5 chose model 4 and 1 chose model 10.

Children drew a "normal" opening with a median of 70°, rather on the large side, but almost all children drew the extreme open and closed positions accurately. However, only 2 of the 12 children drew abstract diagrams using 2 intersecting lines.

To summarise, the crossings context would seem to be easy to represent abstractly, probably because the 2 arms of the angle are physically present. However, the fact that the 2 arms clearly cross each other could be a major obstacle to recognising the similarity to other angle contexts; the fact that the arms are equal in length might also be an obstacle.

Bends

The bends context proved to be very difficult. Despite the physical action of having to slow down and turn the toy ambulance carefully in order to negotiate the sharper bends, children were often unable to compare bends visually. Thus, only 27% of responses correctly identified the hardest and the easiest bends. Unfortunately, only half the children were asked how they could tell whether a bend would be hard or easy. Of these, 3 equated difficulty with the amount of turning needed to get around the curve whereas 3 took a more detached perspective: *Some are sharp and some are not; the easier ones are wider; the easy one is just a bit of a curve.* Students had no difficulty naming other examples of things which bend like a road, but their responses indicated an ambiguous interpretation of bending. Of the 36 examples given, only 10 (mostly parts of the second easiest, angle which differed from the correct choice by only 10° and 20° respectively.

We must conclude that it is difficult for young children to recognise angles in the "road bends" context. Objects which more naturally bend to form 2 clear straight segments (arms and legs, for example) might be more suitable. In fact, the term "bends" appears to cover 2 contexts, only one of which is appropriately modelled using angles.

SIMILARITIES BETWEEN CONTEXTS

We are now in a position to summarise the characteristics of the 6 contexts investigated which could help or hinder recognition of angles (see Table 1). The crucial attributes of the general angle concept are taken to be 2 lines meeting at a vertex with an angular relation between them. In each context, specific features render some of these attributes obvious and obscure others.

Table 1: Distinguishing characteristics of 6 angle contexts

Context	Helping characteristics	Hindering characteristics
Turns	• Vertex of angle is physically obvious • Effect of angle size is intuitively familiar • Size of large angles is intuitively familiar	• Current direction must be represented by a line • Initial direction must be represented by a line • Size of small angles involves fraction concept • Turn is difficult to draw
Slopes	• One arm of angle is physically obvious • Effect of angle size is intuitively familiar	• A reference direction must be imagined
Crossings	• Vertex and both arms are physically obvious • Angle size is easily represented	• Lengths of arms must be ignored • Extension of arms beyond the vertex must be ignored
Bends	• Both arms are physically present	• Curved part of bend must be ignored and a vertex imagined • Sudden and gradual bends must be differentiated • Physical turning must be related to visible angle
Rebounds	• Effect of angle size is intuitively familiar	• Path of moving object must be represented by a line • Vertex must be identified with the point of rebound
Corners	• Vertex and both arms are physically obvious • Effect of angle size is intuitively familiar	• Arms of angle must be detached from region between them • Any rounding at the vertex must be ignored

It is obvious from Table 1 that the 6 selected contexts differ widely. If abstraction theory is correct, learning a general abstract concept of angle could be an enormous task. However, Year 2 students appear to have already taken the first step in the hypothesised abstraction process by recognising similarities between subsets of angles situations, as show by their responses to requests to name "things like this" in the 6 contexts. Suggested examples rarely fell outside the contexts proposed, although children's responses have in some cases forced a refinement of the researcher's original definition of each context. In other words, most of the children had already formed separate concepts of turns, slopes, crossings, bends, rebounds and corners. But did they see any relation between these 6 contexts?

Fifteen students reported angle-related similarities between the 2 contexts presented to them; Table 2 shows the number (out of 4) for each pair. The table includes 4 students who said *They both slope* even though it was not clear that they conceived slope as a relation between 2 lines; but it excludes 2 students who claimed a similarity but were unable to describe it and 9 who gave non-angle related explanations such as *A doll turns in a circle and circles are bends* and *You could use the scissors to measure how wide the road is.* Of the 15 students in the table, 5 showed only how one situation could be seen in the other (e.g. *The hill turns like the doll* and *The scissors could be a slope if you put it like this* [one arm sloping, one arm vertical]. The other 10 mentioned a common property such as *[Bends and scissors] both cross, both open and close, both open really wide* and *Corners and bends are the same except corners are pointed, bends are curved.* Of the 15, only 2 stated an analytical similarity; one said of bends and corners *They're both 2 lines coming together* and the other put the ends of her 2 index fingers together and said about slopes and corners *They all go up like that.*

Table 2: Numbers of students reporting angle-related similarities between pairs of contexts presented

Number of students	Pairs of contexts
0	Turns/rebounds; crossings/rebounds
1	Turns/bends; bends/corners; rebounds/corners
2	Turns/slopes; crossings/bends
4	Slopes/crossings; slopes/corners

With only 4 students per pair it is difficult to make any definite conclusions, but the data are suggestive. The slopes, crossings and corners contexts seem to be the most closely connected; all students recognised the angular similarities between the pairs presented (crossings and corners were not presented together). Turns and rebounds were rather difficult to relate to each other or to the other contexts, whereas bends were loosely related to the slopes-crossings-corners complex. These connections seem reasonable in that slopes, crossings, corners and bends involve physically present lines whereas both angle arms have to be constructed in the turns and rebounds contexts (and in 2 very different ways). Bends may be conceived differently from slopes, crossings and corners because the effort needed to make or negotiate the bend is inversely related to the (interior) angle formed; a similar difficulty relating "turtle turns" to angles is frequently reported in LOGO investigations (Clements & Battista, 1992).

The data can now be used to test the basic hypothesis of abstraction theory. The hypothesis would predict, for example, that all students who used abstract models appropriately (i.e. modelling angular relations between 2 lines) would also recognise a corresponding similarity between their 2 contexts. In fact, only 7 out of 36 children consistently used either the same abstract model (4 children) or 2 different models appropriately. Of these, 6 were able to describe an angular correspondence between the contexts and one seemed to recognise a similarity (between rebounds and crossings) but was unable to express it: *I think so — if it* [the scissors] *was bigger — if it was down there — it's almost like scissors.*

The basic hypothesis would also predict that no student who failed to recognise any angle-related similarity between 2 contexts would consistently use the abstract models appropriately. In fact, only 2 of the 19 students who denied any similarity or described a non-angle related similarity had represented both contexts as a relation between 2 lines — but both of them failed to use the abstract model for one context to represent the other context in that way.

These two results seem to provide strong if limited support for the basic hypothesis of abstraction theory. Table 1 shows that each context has its own specific features, and it seems obvious that these can only become irrelevant when the similarity to a different context is perceived and abstracted.

It could be argued that the abstract models were not spontaneous representations; the fact that children could use them does not mean that they would naturally interpret situations in that way. It might be more valid to look at children's drawings as indicators of how they spontaneously represent situations. However, children made very few abstract drawings; 9 out of 12 drew standard angle figures to represent tile corners, but each of the other five contexts evoked no more than 2 such drawings. This was partly the fault of the instructions which asked children to draw the model in front of them rather than a general representative of the given context.

CONCLUSIONS

The abstraction model of concept development has proved most productive in directing research towards an investigation of physical contexts as they are perceived by children before they form a general angle concept (and before angles are taught formally in school). The salient general and specific features of 6 common contexts have been identified, and the basic hypothesis of abstraction theory has been supported.

The combined data suggest that, for 7 year olds:

- turns, crossings and corners are most easily modelled abstractly,
- corners are most readily drawn abstractly, and
- slopes, corners and crossings are most easily related.

By comparison, bends and rebounds are difficult to treat abstractly. These findings confirm that, as in common curriculum practice, corners constitute a good context for early instruction on angle, but suggest that a more general concept could easily be conveyed by looking for angles in slopes and crossings at the same time. Turns, bends and rebounds might well form topics to be taught separately later.

Research is currently under way to replicate the study among 9 year olds, and it is hoped to extend it to cover all 14 angle contexts later. Meanwhile, studies applying the abstraction model to further mathematical concepts (e.g. fractions and multiplication) are being designed.

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