A PROCESS-ORIENTED PREFERENCE IN THE WRITING OF ALGEBRAIC EQUATIONS

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This paper considers the difficulties which many students, across a wide spectrum of the ability range, experience in correctly translating word problems into mathematical symbols. There is particular reference to problems which give rise to linear algebraic equations, and it identifies the process-oriented ordering preference which procedural students have been found to display when attempting such translations. It also shows that for students who have not fully encapsulated the concept of equation there is an apparent relationship between this preference and the well documented reversal error for translations of word problems such as the 'students and professors' problem. It suggests that for some of these students, as the structural difficulty of translation problems increases, then the students' tendency to revert to the process-oriented ordering preference also significantly increases, and that this has a primary effect on the occurrence of syntactic translation, resulting in the reversal error.

INTRODUCTION

It is increasingly clear that mathematical learning can no longer be limited to the acquisition of skills but must be seen in terms of the thinking processes which students engage in. Their cognitive processes and structures greatly influence their progress in understanding, and hence succeeding in, mathematics. My research in the early learning of algebra (Thomas, 1988; Tall & Thomas, 1991) has identified qualitative differences in thinking between students who do well in this branch of mathematics and those who do not, and it is one such difference and its implications which are the subject of this paper.

DIFFICULTIES WITH WORD PROBLEMS

One area of mathematics which has produced much discussion in the literature has been the cognitive difficulties associated with mathematical word problems. One of the more significant pieces of research in this area has involved what is known as the 'students and professors' problem. This problem was part of a study of the translation of word problems into mathematical equations and involved an investigation of 150 calculus-level students' attempts to answer the problem :

> "Write an equation for the following statement: 'There are six times as many students as professors at this university.' Use S for the number of students and P for the number of professors."
> (Clement, Lochhead and Monk, 1981, p.288).

The results of the investigation showed that 37% of the students answered incorrectly and that two-thirds of these gave the answer as $6S = P$, rather than $S = 6P$, with this transposition error being described as the *variable reversal error.* The possible reasons behind such an error have been the subject of much debate. It is now commonly accepted (see summary in Laborde, 1990) that the attempt to translate directly from the words in the problem into the symbolic notation of mathematics is a fundamental cause of the error. This *syntactic translation* error has also been labelled the *word order matching error*, and has been contrasted by Herscovics (1989) with *semantic translation.* It has proven very resistant to attempts to overcome it. In a recent study related to this error and the possible cognitive processes underlying it, Stacey & Macgregor (1993) have come to some useful conclusions. The results of their study are worthy of note in that they showed clearly that the published causes do not adequately explain the reversal error. They found, for example, that in response to the item :

z is equal to the sum of 3 and y. Write this information in mathematical symbols.

among other answers, 66 students wrote $3 + y = z$, 9 wrote $y + 3 = z$ and 36 wrote $3y = z$. They concluded that these could not have been produced by a direct left to right translation and the equations must have come from a random cognitive model:

> The variety and form of students' responses leads us to infer some properties of their cognitive models and to postulate that information from these cognitive models can be retrieved in any order. Such a retrieval process would explain the apparently random choice of responses that match or do not match the word order (Stacey & Macgregor, 1993, p. 228)

My evidence partly supports this view. However it seems that the mathematical maturity of many children is insufficient for them to have developed cognitive models where information can be retrieved in any order. Such flexible access seems constrained by their previous experiences.

PROCEDURAL VERSUS PROCEPTUAL THINKING

Much of the symbolism used in mathematics inherently carries for the mathematician the dual roles of process and concept. This distinction between the usage of symbolisation to stand for a process and a concept or conceptual structure depending on one's point of focus is clearly an important one mathematically and we call such an entity a *procept*, and the ability to be able to switch ones focus between the dual roles of the symbols as necessary, *proceptual thought.* (Gray & Tall, 1991; 1994). Such procepts and particularly the versatility of the mathematician to vary his/her focus from one aspect of the symbolism to the other in thinking appears to be important in success in mathematics (compare e.g. *operational-structural* conception differences, Sfard, 1987). The failure of many students to think proceptually appears to be due to their inability to perceive structure and form among mathematical symbols and to alter their perspective in order to view the symbolism conceptually. Perceiving the symbolism as representing a process seems far more natural and accessible to the mind of many mathematics students. This is understandable since many of their early mathematical experiences are congregated around arithmetic processes which result in well-formed products. One of the outcomes of this process-product formulation in arithmetic is that the '=' sign acquires a very specific meaning of 'makes' or 'here is the answer to the process carried out', and this view is carried over into algebra (Kieran, 1981), even persisting among college students (Mevarech and Yitshak, 1983). An example of the difficulties students experience is the *process-product obstacle* (Tall & Thomas, 1991), which arises in attempts to understand the equivalence of algebraic expressions represented by, for example, the notations, 2(a+b) and 2a+2b, where these equivalent expressions represent totally different processes.

The wide variety of, and stress on, procedures in the students' early learning of mathematics, I believe, leads to the dominance of this process-led mode of thought, and a lack of experiences designed to promote the global view of a mathematical situation or symbolism tends to stunt the students' ability to develop *proceptual thought.* It appears that some students' procedural inclination makes the step from the understanding of algebra as a procedurally based system of operations with products to a domain of concepts and structures a big leap in the cognitive view. I have begun a theory of *cognitive integration* (Thomas, 1988) which espouses the creation of mathematical schemas in both hemispheres of the brain through an increased use of visual imagery and multiple linked-representations of mathematical concepts (e.g. Kaput, 1989). Such *cognitive integration* of the processing power of both hemispheres would make available to the conscious and sub-conscious reasoning minds (Sperry, 1982) both sequential and global/holistic modes of thought in a given context. The goal is to mediate more easily proceptual modes of thinking by increasing the possibility of the student being able to vary their cognitive focus in a given mathematical situation from a sequential, left-to-right process-led perspective to a global/holistic concept-driven mode.

My hypothesis is that the procedural inclination of beginning students of algebra predisposes them to write certain algebraic equations with the process construct on the left side of the equation and the 'answer' or result on the right hand side, instead of the more usual *assignment order* with a subject variable. I call this the *process-oriented ordering preference* for writing equations.

THE EMPIRICAL STUDY

In order to investigate whether a procedural preference was influencing students' thinking in translation problems I designed an investigation comprising a carefully prepared range of linguistically presented equations, with students required to construct algebraic equations from the given forms. It was given to a group of 75 year nine high school students at a selective school, where the students were roughly in the top 35% of the ability range and a second group of 128 university undergraduates training to be teachers of children aged 4-12. Some of the results of the survey are summarised in Table 1.

Table 1: Translation Responses for Additive and Multiplicative Problems

Item	Responses	School	University
1. y is equal to x plus four	$y=x+4$ $\{y=(x+4)\}$	65	107
	$x+4=y$	9	15
	$4+x=y$	0	2
	$y=x+4$ or $x+4=y$	1	3
	other	0	1
2. w is equal to the sum of 3 and n	$w=3+n$ $\{w=(3+n)\}$	52	91
	$w=n+3$	4	2
	$3+n=w$ $\{(3+n)=w\}$	10	30
	$n+3=w$	2	2
	$w=3+n$ or $3+n=w$	0	1
	other $\{$e.g. $3n=w$, $w=3n\}$	7	2
3. A school has v girls and t boys. There are ten more girls than boys. Write an equation relating v & t.	$v=t+10$ $\{v=10+t, t=v-10\}$	41	33
	$t+10=v$ $\{10+t=v, v-10=t\}$	24	44
	$t=v+10$ $\{v=t-10\}$	1	4
	$v+10=t$ $\{10+v=t, t-10=v\}$	7	24
	others $\{$e.g. $10t=v$, $t=10v\}$	2	23
4. m is 5 times n	$m=5n$ $\{m=n\text{x}5\}$	49	89
	$5n=m$ $\{n\text{x}5=n\}$	19	26
	$n=5m$	1	0
	$5m=n$ $\{m\text{x}5=n\}$	5	13
	Other	1	0
5. A record by Take That is h minutes long. A record by Kriss Kross is g minutes long. The Take That record is three times as long as the Kriss Kross. Write an equation relating h and g.	$h=3g$ $\{h=3\text{x}g, h=g\text{x}3\}$	33	45
	$3g=h$ $\{3\text{x}g=h\}$	31	50
	$g=3h$ $\{g=h\text{x}3\}$	1	5
	$3h=g$ $\{3\text{x}h=g, h\text{x}3=g\}$	9	25
	other	1	3
6.A band makes four times as many asingles as albums. It makes q albums and z singles. Write an equation relating q and z.	$z=4q$ $\{z=q\text{x}4, z=q4\}$	20	13
	$4q=z$ $\{4\text{x}q=z, q\text{x}4=z\}$	23	59
	$q=4z$ $\{q=4\text{x}z\}$	12	13
	$4z=q$ $\{4\text{x}z=q, z\text{x}4=q\}$	20	38
	other	0	5

Analysing these results I found that in simple translation problems, with an operation of addition, such as items 1 and 2 in the tables above, the reversal error does not exist for either group of students. This agrees with the findings of Stacey & Macgregor (1993) who reported that the error was very rare in such examples. However the *process-oriented ordering preference* was present (Table 2), demonstrating this to be a distinct factor in the thinking of early students of algebra, and one which persists even for more mature students of mathematics. The variables x and y are very familiar to students in the *assignment order*, $y = mx + c$, yet in spite of this the *process-oriented ordering preference* is present in item 1. The following unsolicited comments written on their questionnaires by two of the university students confirm this preference. Alongside her response of 4+3=7 one wrote writing 7 = 3 + 4 may confuse children

another who showed the preference in *every* response, wrote alongside them

got to have sum before answer in our mind

This second comment seems to be a clear example of process-led thinking.

Table 2: Proportions of Students Showing the Process-Oriented Ordering Preference In Basic Translation With Additive Operation

	Item 1	Item 2
Year 9 students	0.12	0.16
University students	0.133	0.25

A *process-oriented ordering preference* (although not reported as such by them) was also shown in the data recorded by Stacey & Macgregor (1993), with 0.435 of their sample displaying it on a corresponding question. This may be an indication that it is even more prevalent among students of lower mathematical ability than those in this present study.

As the amount of detail given and the order of the data make the translations more difficult, the incidence of the *variable reversal error* increases. However, we also see that the proportions showing the *process-oriented ordering preference* increased too. When we compare the proportions for items 2 and 3 we see that in each instance, and for each group of students, there is a significant increase in the occurrence of both the reversal error and the *process-oriented ordering preference*. These results also show that, not only does the reversal error persist in students thinking even as far as undergraduate level, but that, more surprisingly, so does the *process-oriented ordering preference*. Furthermore, each significantly increases in occurrence as the complexity of the additive word problem increases.

Table 3 : The Increase in the Proportion of Students Showing Reversal Error and Ordering Preference From Basic Additive Word Problem Translations to More Complex

	Reversal Error				Ordering Preference			
	Reversal Error Item 2 Only	Reversal Error Item 3 Only	z	p	Ordering Preference Item 2 Only	Ordering Preference Item 3 Only	z	p
Year 9 students (n=75)	0	0.107	2.83	<0.01	0.080	0.467	3.31	<0.001
University students (n=128)	0	0.291	6.08	<0.0005	0.078	0.453	5.82	<0.0005

When we consider the three questions which involve multiplication of the variables in the final equation, namely items 4, 5 and 6, we first note that these are in increasing order of difficulty due to the complexity of the word problem format. The first may be completed by *syntactic translation,* but even here we see in student responses both the reversal error and the *process-oriented ordering preference*, and it seems that the reversal error is more likely to occur when the operation is multiplication than when it is addition. Item 5 is considerably more difficult due to the context, but can be completed by *syntactic translation*. However, item 6 has both the complex structure and context and a word order requiring *semantic translation*. These last two word problems, and especially the final one, are of a high order of difficulty for this class of problem, and correspond to the students and professors problem. In these questions we find, not surprisingly, a significant increase in the proportions making the variable reversal error.

Table 4 : The Increase in the Proportion of Students Showing Reversal Error and Ordering Preference From Direct Ordering to Complex Ordering Multiplicative Word Problem Translations

	Reversal Error				Ordering Preference			
	Reversal Error Item 4 Only	Reversal Error Item 5 Only	z	p	Ordering Preference Item 4 Only	Ordering Preference Item 5 Only	z	p
Year 9 students (n=75)	0.082	0.151	1.21	n.s	0.135	0.338	2.54	<0.05
University students (n=128)	0.063	0.203	3.09	<0.01	0.047	0.352	5.46	<0.0005

What is of great interest too is that the incidence of the *process-oriented ordering preference* also increased significantly between the basic type of problem and the more complex, contextual problem and that, for the university students at least, it also increased significantly when the ordering of the words in the the original makes the structure of the contextual problem even more difficult, as between items 5 and 6.

Table 5 : The Increase in the Proportion of Students Showing Reversal Error and Ordering Preference From Syntactic Ordering to Complex Ordering Multiplicative Word Problem Translations

	Reversal Error				Ordering Preference			
	Reversal Error Item 5 Only	Reversal Error Item 6 Only	z	p	Ordering Preference Item 5 Only	Ordering Preference Item 6 Only	z	p
Year 9 students (n=75)	0.014	0.311	4.49	<0.0005	0.095	0.135	0.73	n.s.
University students (n=128)	0.117	0.273	2.83	<0.01	0.086	0.234	2.97	<0.01

Looking at the interaction between the reversal of the variables error and the occurrence of the *process-oriented ordering preference* we see that (in Table 6) for the year nine students the results seem to give evidence of a significant association between the occurrence of the *process-oriented ordering preference* and the reversal error.

Table 6 : Proportions Showing The Interaction of the Reversal Error and the Process-oriented Ordering Preference for Year 9 and University Students

	Year 9 Students				University Students			
Item Number	Ordering Preference only	Reversal Error only	χ^2	p	Ordering Preference only	Reversal Error only	χ^2	p
3	0.329	0.014	7.46	<0.01	0.419	0.038	7.34	<0.01
4	0.257	0.014	7.72	<0.01	0.203	0	33.0	<0.0??
5	0.419	0.014	6.02	<0.05	0.40	0.040	8.96	<0.01
6	0.307	0.160	0.61	n.s.	0.480	0.106	0.99	n.s.

The conflicting interference of two competing cognitive influences, the ordering preference and the left to right sequential processing of the natural language make a serious obstacle to a correct formulation of the problem in terms of its variables, particularly in examples where the structure of the problem is more complex. The evidence seems to indicate that the harder the word translation problem is the more likely some students are to revert to their more familiar procedural thinking, and syntactic translation apparently becomes a major factor only in the most difficult examples.

CONCLUSIONS

It seems that as problem statements move from being syntactically equivalent to the algebraic assignment formulation to more complex statements, the student responses increase in their use of process-oriented statements with the operation on the left and the result on the right. In cases where the syntax becomes too complicated to support a straight translation and the words are used in a way which does not encourage the use of letters as objects, errors which reverse the roles of letters still occur. Such errors may occur because of the cognitive complexity rather than a specific syntactic misconception. It appears that experience in the domain of arithmetic with its emphasis on carrying out a process to obtain an 'answer' or product gives some students a predisposition to a *process-oriented ordering preference.* Their inability to utilise symbolisms which imply holding a process in abeyance or accepting the result of a process as yet unspecified, is a serious obstacle to a meaningful understanding of algebra. These results, combined with the conclusions of other studies, suggest at least two major factors influencing student errors in questions involving the construction of equations from word problems. First, well-formed cognitive processes based on the preference for the procedure-output model, which I have called the *process-oriented ordering preference,* and second the use of *syntactic translation,* when the written form of the word problem is relatively complex, leading to the *word-order matching error.*

We have also seen evidence that the first of these two factors is persistent in the thinking of students, even being present at the undergraduate level. Further research to investigate the extent to which each of the above two factors is involved in influencing the outcome of students responses and the level of interaction between them will be necessary.

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