Organisational features of geometric knowledge

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Despite the increasing need to understand the nature of mathematical knowledge that students develop, we have limited tools to examine or assess this knowledge. In this paper we report the use of five indicators that could provide useful information about the nature of students' geometric knowledge. These indicators are then used to examine the relationship between structural quality of students' geometric knowledge and their problem-solving ability.

Introduction

Investigations into students difficulties with mathematical problem-solving have identified a number of variables that are strongly related to high-levels of performance. Recent research in this area is beginning to examine the relationship between the nature of mathematical knowledge that is constructed by the student, the cognitive actions exhibited by the solver, and the moves made during a solution attempt. Specifically, mathematics educators and cognitive psychologists are developing explanations about how the quality of mathematical knowledge drives the accessing and exploitation of that knowledge during the solution process (Shoenfeld, 1985). Some researchers have begun to examine the link between the quality of the students' knowledge bases and their success in problem solving (Prawat, 1989; Lawson & Chinnappan, 1994). The broad rationale for this research is that , among other factors, mathematical knowledge bases that are effectively organised and structurally more complex will facilitate more effective activation and use of knowledge during problem solving (Sweller, 1988)

At this time there is limited data on ways of gaining information about the quality of knowledge structures. Research in cognitive psychology suggests that one way to gain insight into the structural qualities of domain knowledge is to examine the rate at which a particular item of information is accessed when a cue is presented. The assumption underlying this approach is that items of information that are accessed faster by the students will be those items that are better connected within the student's knowledge base. This is *not* assumed to be a comprehensive indicator of the quality of the knowledge base. Access time provides only one indicator of knowledge connectedness. However this time data can be used alongside other indirect indices of the knowledge structure to provide a basis for judgements of quality of that structure. If the assumptions about the relationship between speed of knowledge access, structure of mathematical knowledge and knowledge utilisation are valid, one would expect high-achieving students of mathematics to access mathematical knowledge more rapidly than their low-achieving peers. In this study we attempted to generate data in support of the above prediction.

Another indicator is the quantity of problem-relevant information that a student is able to recall spontaneously. This is an indicator of what might be able to be accessed by the student during a problem attempt. Problem-relevant knowledge components that can be recalled without assistance can be seen as having high strength and or high states of activation in Anderson's two concept theory of memory (Anderson, 1990). This implies that the student's knowledge structure must be in an effective state of organisation.

A related indicator identifies the amount of assistance required by a student to access problem-relevant knowledge. If a student is unable to access a particular knowledge component spontaneously, one could provide assistance in the form of hints to facilitate the retrieval of that information. Those students who require greater levels of assistance for the access task are argued to be working from a knowledge base that is less well connected. We explored this possibility by devising a task involving the provision of hints to students and examined the extent of assistance needed by students.

The degree of use or exploitation of the knowledge is a further indicator of the state of the knowledge structure. The effective utilisation of the accessed knowledge could involve the retrieval of further knowledge from LTM that has previously been connected in some meaningful way. Hence, knowledge that can be accessed and then used on a range of problems is argued to be more effectively organised. This line of reasoning suggests that tasks that explore knowledge extension could be utilised to provide further indices of the quality of knowledge base.

In the present study we provide information about the organisational quality of students' geometric knowledge using these four sets of indicators - time, spontaneous retrieval, assisted retrieval, and use. We explain the procedures used to gather information for each indicator and report on the relationship between these indicators and student problem solving performance.

Method

Participants

The participants were 36 Year 10 students from a private college in metropolitan Brisbane. In this college students were streamed into different classes on the basis of their performance in Year 9 and Year 10 mathematics tests. The college curriculum requires that all students complete a topic involving trigonometry and geometry during Years 8, 9 and 10. At the time of year of this study all the students had completed this topic. High achieving students (HA: n = 18) came from the upper two Year 10 streams. The low-achieving students (LA: n=18) came from the three classes of the lower streams.

Procedure

All students were interviewed individually in two sessions, each lasting 60 minutes. During the first session, students were required to complete four tasks: the Free Recall Task, the Problem Solving Task, Geometry Components Task and the Hinting Task. The Free Recall Task required students to identify known geometry theorems and formulae . Students were asked to indicate this by written statements and diagrams. The Problem solving task (PST) consisted of four plane geometry problems which can solved by the use of theorems and formulae that are taught in the first three years of the high school mathematics curriculum.

The Geometry Components Task was developed in order to examine students' knowledge of parts of geometric figures and of the theorems or rules that are represented by these figures. Students were given five figures, each on an index card. Students were required to identify the parts of the figure(GCTForms) and to produce the required theorem (GCTRules).

The Hinting Task required the development of a series of hints on the basis of a commonly adopted solution path for each of problems 1, 3 and 4. When students failed to produce the correct solution for one of these problems, they were required to attempt to solve that problem with the help of hints given by the investigator. If necessary students were given further hints, with the final hint being a give away hint that showed a solution.

During the second session, students were required to complete three tasks: the Recognition and Timing Task, the Geometry Application Task, and the Geometry Elaboration Task. The Recognition and Timing was based on Hypercard software which recorded the time taken by a student to correctly identify a particular geometric form, initially without any assistance. This task involved students in identifying the names of selected features of geometrical forms that were displayed on the computer screen. The features were located on buttons and had associated fields into which students could enter labels for each feature or form. The Hypercard figures were developed so as to capture commonly taught geometric schemas in the classroom, such as a right-angled triangle and its properties.

The Geometric Application Task required students to produce an example of use of a given set of five theorems or formulas. If students were unable to produce an example, they were provided with simple problems which they had to solve. Completion of Geometric Elaboration Task involved the investigator presenting a pair of theorems of formulae to the students. Students were required to generate a problem that involved use of both the given set of geometric information in its solution space.

Results

Free Recall Task

If high-achieving students develop more effectively organised geometric schemas we should expect them to be able to retrieve more extensive bodies of knowledge in a free recall situation. The results of this task will not isolate the reason for this outcome: Better recall performance could reflect the existence of either more extensive available knowledge or more effective recall of available knowledge. However, this task does provide one indication of the state of knowledge of students. Knowledge components that can be retrieved easily in this task can be seen as having high strength/activation values. The results of this comparison showed that the free recall for the HA students (mean 10.83; SD 4.48) was significantly higher than that of LA students (mean 7.06, SD 5.78), $t(34) = 2.19$, $p<0.02$.

Problem Solving Task

Student performances were scored as follows: 0-incorrect solution, 1- partially correct solution, 2- correct solution. The results of this analysis showed that students from the high-achieving group performed significantly better than those in the low-achieving group. The respective means and standard deviation were; HA : M= 4.72 (2.02), LA: M= 2.39 (1.91): $t(34) = 3.55$, $p<0.001$. In this task students have more support for their accessing of knowledge than is provided in the free recall task, since the problem statements and diagrams provide cues that can be used to search memory. The greater success of the HA group indicates that these students were able to retrieve more of the problem-relevant knowledge.

Geometry Components Task

Group performances on this task revealed that HAs did not differ significantly from the LAs in the identification of features of the given figures (GCT-Forms). In this task the students again have available cues from the diagrams and this level of support appears to assist them to improve the level knowledge access over that seen in the Free Recall task. However, there was a significant difference between the groups in identification of theorems/formulae (GCT-Rules) which represent knowledge clusters that show the link between the various parts, t (29) = 3.42, $p<0.01$. This suggests that the knowledge bases of the HA students allow for more effective connection between discrete knowledge components.

Hinting Task

In the Hinting Task explicit assistance was provided to students in a manner that was designed to identify the functional availability of knowledge components in this area of geometry. This task is important because it allows stronger inferences about the locus of the students' access performance. The graded hints provide the students with cues for searching of knowledge and so decrease the load imposed by the search process. If students are unable to access relevant knowledge after provision of the final hint we can have greater confidence in the inference that knowledge availability is the problem. If provision of the earlier hints result in successful access the organisational state of knowledge would appear to be less effective than it could be. Analysis of the number of hints required showed that the HA required fewer hints than the LA students, the means and standard deviations being; HA: M=13.03 (9.74), LA: M= 21.06 (10.25), t (34) = - 2.40, $p<0.05$.

Geometric knowledge access time

The Recognition and Timing Task generated the time taken by each student to retrieve geometric knowledge components that were required to solve the four problems given in the PST. The software designed for this task provides the times at three levels: level 1, 2 and 3. At Level 1, students are not provided with any assistance by the computer program. At Levels 2 and 3, students are given increasing amounts of assistance. In this study we report access times for Level 1 only. Group access times were analysed in two ways. The first analysis focused on a range of components that are covered in the school mathematics curriculum which could have been used in the solution of the problems in the PST. Mean access times for the individual components are shown in Table 1. HA students were able to retrieve most of the components under consideration faster than their the LA peers. The figures in the middle column for each group indicate the number of students who successfully accessed the component without assistance. For 11 of the 13 components the HA group had faster access times, seven of these being significantly faster than those of the LA group.

The other column for each group shows the number of students in the group who were able to identify the component when provided with the assistance available in Levels 2 and 3 of the timing program. The effect of assistance was greater for the LA group, with more of these students showing benefit from the cues made available in the program. For any student not included in these two columns we argue that the component is not functionally available.

Table 1: Access times for a set of Geometric schemas

Geometry Knowledge Components	High-achievers (n=18)				Low-achievers (n=18)				p
	Mean	SD	CL1	A/H	Mean	SD	CL1	A/H	
<u>Right-angled Triangle</u>									
Hypotenuse	8.05	2.56	16	0	11.67	9.27	11	6	
Right-angle (90deg)	15.35	15.35	17	0	12.16	10.88	16	1	
<u>Figures</u>									
Right-angled triangle	9.38	4.26	14	0	12.89	9.89	11	5	
Straight line	7.27	2.42	16	0	16.93	12.06	14	2	*
Equilateral triangle	6.95	3.81	16	0	14.48	12.34	11	5	*
<u>Tangent-radius Figure</u>									
Tangent	8.64	4.46	16	0	12.66	7.28	11	5	*
Right-angle (90deg)	7.62	3.78	15	0	15.65	11.45	10	4	*
Circle	6.53	3.10	17	0	9.92	6.68	12	3	*
Radius	6.99	3.00	17	0	13.95	14.22	13	3	*
<u>Trigonometry</u>									
Tangent ratio	4.17	1.39	16	0	4.48	3.23	09	4	
Cosine ratio	6.41	5.83	16	0	5.60	1.92	10	3	
Sine ratio	10.47	4.33	15	1	4.59	5.39	08	5	*
<u>Supplementarity</u>									
Sum of angles on a straight line	7.90	1.95	17	0	1.86	7.13	16	1	*

CL1: Correct at Level 1; A/H: Accessed with hints; * $p<0.05$

Geometry Application Task

Students were required to provide illustrations of use five geometric theorems or formulae. The mean score for the HAs (M= 20.67, SD=5.57) was significantly higher than that of the LA students (M= 16.89, SD=3.80), t (34) = 2.38, $p<0.05$.

Geometric Elaboration Task

In this task students had to show the various ways in which a given pair of theorems or formulae could be associated in a geometrically meaningful manner. Unlike GET, students were not given any assistance in the form of diagrams. Students had to construct their own figures. This analysis showed that the high-achieving students were able to show a greater degree of knowledge extension than the LA students. (HA: M = 8.06, SD = 4.09), (LA: M = 4.49, SD = 2.89), t (34) = 3.10, $p<0.01$.

Correlations

Correlations were computed between the students' solution scores and their scores on the other measures. The correlation coefficients are shown in Table 2. The pattern of relationships among the measures were largely as expected. Students' solution scores (Sol) on the PST were strongly related to the measures that were used as indicators of access and connectedness. We see this in the correlations of solution score with number of hints and with rules, application and elaboration measures. Correlations of problem solving success with the other two measures were negligible. The ability to identify features and forms as discrete components was not a good predictor of success. Access time at Level 1 of the Recognition and Timing task was only weakly associated with problem solving outcome.

Table 2: Correlations between problem-solving performance and indicators of geometry knowledge organisation

	Sol	Hints	Forms	Rules	Appln	Elab	Acctime
Sol	1.00	-0.77**	0.024	0.55**	0.49**	0.44 **	0.14
Hints		1.00	-0.11	0.51**	-0.40**	-0.44**	0.34
Forms			1.00	0.58**	0.32*	0.47**	-0.02
Rules				1.00	0.40*	0.41*	-0.25
Appln					1.00	0.55**	-0.15
Elab						1.00	-0.14
Acctime							1.00

* $p<0.05$; ** $p<0.01$

Sol: solution score; Hints: frequency of hints; Forms: GCTForms Rules: GCTRules; Appln: Geometry Application score; Elab: Geometry Elaboration score; Acctime: access time

Discussion

The differences in problem-solving performance of the two groups in this study are not surprising: Students identified as higher achievers in their mathematics classes achieved higher scores than lower achieving peers when asked to solve a set of geometry problems. What was of interest here is what characteristics of students' knowledge were associated with this pattern of results. We set out to identify indicators of knowledge organisation that were related to problem solving performance. In general the indicators chosen for this purpose showed quite strong relationships with successful outcome.

The HA students were able to access a wider body of knowledge of geometry facts and theorems in a free recall situation than the LA group. The results of the Geometry Components Tasks suggested that this difference was most pronounced in the more complicated parts of this knowledge, in the recall of theorems. Other results suggest that this difference in access was not just the result of a difference in availability of knowledge. The results of the Hinting task showed that the LA students required more assistance to access knowledge relevant to the problem set. This was also apparent in the timing task where a greater proportion of the LA students required assistance in order to identify knowledge geometric forms and rules. The provision of assistance had an impact on a greater number of the students in the LA group. In the cases where this assistance proved effective the students were able to gain access to knowledge that had previously been in an inactive state. For this reason we suggest that the hinting and timing tasks do provide indicators of the organisational state of the students' knowledge bases. The benefit gained from the provision of hints suggests that these procedures that can be used to search for relevant knowledge are not as effective as they might be. Training in use of search procedures might help these students to access a wider body of knowledge.

The results of the Application and Elaboration Tasks also address the issue of knowledge organisation. In these cases it is not so much the influence of search procedures that is of concern, but the state of connectedness of knowledge. Students who scored highly on these tasks we argue show evidence of being able to activate wider networks of knowledge. These wider networks reflect the stronger links between related knowledge components.

There were strong relationships between students' scores on problem solving and the following indicators of knowledge organisation: hints, geometric components, geometric application and geometric elaboration. There was not a strong relationship between access time and the level of problem-solving performance. The absence of significant relationship between the access time and solution score could have been due to the fact that we have considered students' access times at Level 1, i.e. the level at which students were not given any assistance. We have not yet examined this relationship at levels 2 and 3, the levels at which students are provided with some assistance. Also we have not looked at the relationships between access times at levels 2 and 3 and the other indicators of organisational quality. Future studies needs to consider these relationships.

The results of this study provide further information about why high-achieving students are able to produce better solution outcomes than low-achieving students. We have argued that, among other factors, the organisational quality of students' geometric knowledge plays a major role in helping them make progress during the solution process. Superior solution attempts of high-achieving students appear to be driven by a geometric knowledge base that is better structured and more extensive in nature than that of low-achieving students. Students' scores on the indicators of knowledge organisation used this study showed that a) they are able to retrieve more of the organised knowledge structures than the low-achieving students and b) given a particular item of knowledge, the high-achieving students showed greater ability at drawing other information than can meaningfully be linked to the given information than their LA peers. Thus, it seems reasonable to argue that underlying the successful problem-solving performance is a knowledge base that is better organised and more extended, supporting the view expressed by Prawat (1988) and Larkin (1979).

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