Analysing Teaching/Learning Strategies for Algebra

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This paper reports developments that have taken place in an ongoing research project since last year's MERGA Conference. Data from two more experimental design stages have been examined. Clarification of the role of concrete analogies to strengthen understanding of arithmetic structures prior to the introduction of associated algebraic concepts has led to the revision of some proposed teaching programs. Further data are to be collected from 22 classes this year. A means of measuring preferred learning styles has been developed.

Relevant Research Perspectives

We have much to learn about the teaching and learning of algebra. As Wheeler (1989, p. 280) points out, 'in spite of the many decades during which it [algebra] has been taught, the pedagogical problems it presents have by no means been solved.' Kieran (1989, p. 53) states 'The challenge to researchers is to devise studies that will push forward our knowledge of how students can come to understand the structure of elementary algebra and algebraic methods.'

We have much to learn about the role of analogies, whether concrete or mental analogies, in the process of teaching and learning mathematics: 'Explanations of concept development and the role of concrete materials in learning have not been the focus of in-depth studies' (Owens, 1994, p. 455). '... little work has been directed towards its role [the role of analogy] in children's learning of basic mathematical concepts and procedures' (English, 1994, p. 213). It is hoped that the project which is the subject of this paper will make some worthwhile contribution in this area.

Perry and Howard (1994, p. 487) document the considerable support given to the use of concrete materials over recent decades. The concrete analogies under investigation in this study are among those recommended in the N.S.W. Years 7 and 8 Mathematics Syllabus which 'recognises the importance of the use of concrete materials as many students in Years 7 and 8 have not yet reached a level of abstract thinking in mathematics' (1988, p. xii). The analogies are explained in Quinlan, Low, Sawyer and White (1993).

Biggs (1992) noted that programs prove to be successful when they include such things as group work, one-to-one interaction between student and tutor, problem-solving emphases, and when: 'abstract, conceptual learning is built on lower-level learning, particularly where a variety of hands-on experiences are used [and] formally taught knowledge is specifically linked to sensory and enactive experiences' (p. 289). Teachers participating in the present study are encouraged to use group work and a problem-solving approach (although these not enforced). Some teachers are to use concrete analogies, 'keeping in mind a multi-modal approach' (Quinlan, 1992, p. 350).

Not all concrete and mental analogies receive commendation. Halford and Boulton-Lewis (1992) devoted 26 pages to discussing 'the values and limitations of analogies in teaching mathematics.' They pointed out that the structure mappings involved in using analogies impose a processing load and that some analogies can increase the learning or memory load rather than reduce the effort needed. Lesh, Landau and Hamilton (1983) found that the choice of concrete model was crucial. Pizzas and cakes as models hindered the learning of addition of fractions, maybe, as Sweller

(1993, p. 10) suggests, because they were 'redundant', whereas a model using egg-cartons was highly successful: 'Virtually none of the students used paper and pencil for the egg addition problems. Characteristics of the materials themselves apparently facilitated the higher rate of correct responses' (p. 289). Ideally, a concrete representation 'mirrors the structure of the concept and the child should be able to use the structure of the representation to construct a mental model of the concept' (Boulton-Lewis & Halford, 1991, p. 37).

The project is urgent and relevant. Insufficient research has been devoted to the analysis of the role of appropriate analogies in teaching and learning. The energies devoted to the project are focusing more and more on

1 the qualities of useful analogies and
2 an attempt to explicate their role in learning.

Recent Research Findings

At last year's MERGA Conference, the author gave a short but detailed report (Quinlan, 1994a) on an experimental design project in which one teacher taught two classes, in November 1993 at School A, to provide data for the comparison of two teaching methods. Both classes were led towards an understanding of equivalent linear algebraic expressions in one variable. Arithmetic examples were used as the basis of understanding for one class and a concrete analogue in the form of an objects-and-containers model was used to assist the process for the other class. No significant differences were detected between the classes before the four periods of teaching intervention but the latter class showed statistically significant advantages afterwards in both attitude and achievement on the subject matter taught. Analyses of variance for the outcomes on the Posttest using Pretest scores as covariates identified the effects of teaching methods as significant with $p \leq .01$, accounting for 62.1% and 55.8% of the variance for attitude and achievement respectively.

Questions arose as a consequence of this experimental support for the use of concrete analogues. Would a similar outcome be found with another pair of classes? What qualities are required of such an analogue and how should it be used if it is to assist learning? Is there any relationship between preferred learning styles and learning with or without concrete manipulatives?

Two further experimental design stages of data collection were reported in Quinlan (1994b). In September 1994 at School B, the topic of indices was taught to two Year 8 classes by their regular teacher. Only one of the classes made use of unit cubes for representing exponential growth, leading to an exploration of the properties of indices. The other class followed a similar teaching sequence and style but without the model. For example, both made sets of cards folded as 'tents' bearing paired expressions for the same sequence of numbers, such as 8 on one side and 2^3 on the other, and used these when exploring the properties of indices. The class using the concrete approach (with the cubes) was significantly better than the class following the traditional approach on Pretest measures of attitude to algebra, overall performance in algebra, and total scores on three test questions (Questions 8, 10 and 14) specifically about indices. These three questions covered aspects such as **Write $a^{3n} \times a^n$ in the form a^n ; Which is larger 50^{150} or 150^{50} ? 2^x or 8^y ?**. They remained significantly better on all these measures in the Posttest. On the question of comparing 2^x with 8^y, analysis of variance showed the effects of teaching methods as statistically significant with $p < .05$, even though the proportion of variance explained was only 19.1%. Over the teaching intervention period both classes improved significantly on Question 7, which asked students to find the values of the algebraic symbols in the following four representations of the number 64: 2^d; $2^2 \times 2 \times 2^a$; $(2^y)^2$; $4^w \times 4^2$. Thus, the data gave support to the view that, although the use of the unit cubes took up students' time, this concrete analogue did not overload their brain capacity and so hinder learning. It was clearly *not* a case of the possibility flagged by Halford (1993, p.381) with reference to using base ten blocks for subtraction, namely, 'if the justification [of analogs] is not understood, the concrete analogs may be worse than useless, because they are extra things to learn, take time to manipulate, and cause distraction.'

In School C, what could be regarded as the worst case scenario for using concrete manipulatives emerged in September 1994. The regular class teacher taught two Year 8 classes about factorization and expansion of first and second degree expressions. Before the teaching intervention, the class to use a traditional symbols approach was markedly superior to the class chosen to use an area model with cut-out rectangles and squares. The students in the latter class had great difficulty at first in relating algebra to the reality expressed by the areas of separate shapes and the composite rectangles which they produced. For instance, when two rectangles had been pushed together to give a rectangle with a length of $y + z$, many wrote yz as the length. The first period was spent trying to iron out some of the difficulties rather than making much progress with factor problems. In the next period the students were instructed to write the area calculations on each piece of cut-out paper squares and rectangles. For instance, $A = 1 \times y = y$ or $A = y \times z = yz$. This strategy reduced the unproductive effort needed 'to attend to several sources of information simultaneously' (Sweller, Chandler, Tierney & Cooper, 1990, p. 189). Despite these difficulties, with four input lessons they recorded significant improvements, with $p \leq .01$, in factorizing $jk + mk$, $k^2 + 2mk$, $6d + 3$, and $6cd + 3d$, and in expanding $j(3 + m)$ and $2j(j + 3k)$. Whether they would have learnt more or less without the area model is not established. The advanced class recorded significant improvement, with $p \leq .01$, in far more difficult questions, such as expanding $(1 + p)(2q + r)$ and factorizing $3xz + yz + 6x + 2y$. Of the 20 students interviewed, 70% said that they really enjoyed using the area shapes and another 10% said they enjoyed them some of the time. Also 70% said that the materials helped them learn, with another 20% saying they helped sometimes. Their attitudes to algebra improved from being significantly poorer than those of their counterparts to almost matching them.

Dempster (1981), after exhaustive examination of many studies, was able to claim that 'memory span is indicative of overall intellectual ability' (p. 65). Moreover, Halford (1993) pointed to the potential disadvantage that some analogues 'can actually increase the learning or memory load' (p. 220). Boulton-Lewis used digit span 'to determine short-term memory' (1987, p.335). Digit-span tests were given to most students in Schools B and C. The digits were in three lists, each growing from 2 to 8 digits. These were assembled from information supplied to the author by Boulton-Lewis (personal correspondence, 16th November, 1994), with the addition of 7- and 8- digit lists taken by the author from a set of random numbers. The numbers were read to the students at about one per second, 'the rate normally used in memory span tasks' (Dempster, 1981, p. 68). The hypothesis that those with larger digit spans would manage the concrete model more effectively than those with shorter spans was not supported by the data from either School B or School C. Efforts to link the digit span result to performance were not successful using correlations with learning gains, analyses of variances, or examining the extreme scores on performance variables. Of interest is the fact that the classes recorded no significant differences on digit spans.

Qualities of Appropriate Analogies

The analysis of scientific analogies given by Gentner (1982) has been found particularly helpful in analysing the qualities of the concrete analogies used. The objects-and-containers model used in School A is discussed here (cf. Quinlan & Collis, 1990a). This concrete analogue provides a sound *base specificity* since the numerical value of any modelled expression is represented by the actual number of tangible objects used. An arbitrary value for an algebraic variable is simply represented by the number of objects placed in a container. *Nodes* which are all aspects of algebraic expressions to be considered in the target algebraic system are characterized as *Variable (F), Sum of Variable and Constant (G), Multiples of Variable (H), Multiples of Expressions (I),* and *Equivalent Expressions (J).* The *clarity* of the model resides in the fact that there is a one-to-one relationship between these target nodes and corresponding base nodes in the concrete model. A variety of algebraic expressions can be modelled with *clarity* and the students soon can explore many *relationships,* thanks to the *richness* of the model. This richness helps students to write great varieties of equivalent algebraic expressions.

As was discussed in Quinlan and Collis (1990b, pp. 445 - 448), the objects-and-containers model has the strengths of commutativity, transferability and isomorphism. The model matches admirably the definition of representations and isomorphism as given by Coombs, Dawes and Tversky (1970, p.11). It can be used to explore mappings back and forth between the algebraic symbolic form and the concrete form because of the *isomorphism* between the structure of the algebra and that presented by the model.

It is becoming clearer that the concrete models used in Schools A, B and C are extremely beneficial in clarifying the structure of the arithmetic required for the understanding of the ensuing algebra. As Halford and Boulton-Lewis (1992, p. 207) point out: 'Elementary algebraic concepts are acquired by using previously learned number concepts based on constants as mental models.' Halford (1993, p. 222) argues that 'A good source for learning algebra depends on having a good mental model of arithmetic relations. To the extent that concrete analogs promote such a mental model of arithmetic, they facilitate acquisition of algebra.'

The point to be made here is that the objects-and-containers model not only clarifies for students the arithmetic structures relevant to the algebra of first degree expressions in one variable. The model then helps them apply their mental model of arithmetic to algebraic structures where arithmetic generalizations need to be understood.

Two possibilities come to mind. First, there could be mutual interaction between the chosen model, arithmetic, and algebra, as depicted in Figure 1.

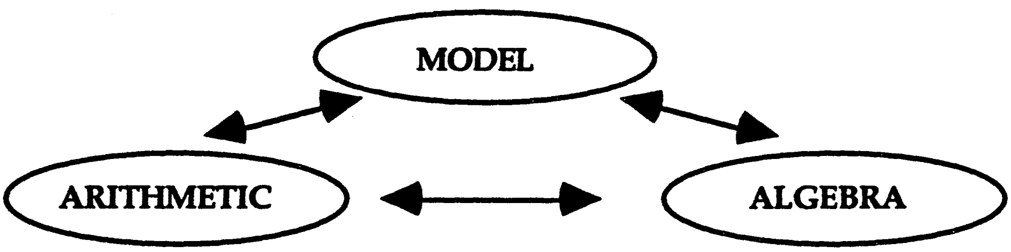

Figure 1. Mutual interaction between arithmetic, model and algebra

Secondly, the model could be an example of what Clement (1993, p. 1244) calls a 'bridging analogy' which assists the learner 'because it is easier to comprehend a close analogy than a distant one; the bridge divides the analogy into two smaller steps that are easier to comprehend than one large step.'

The large step, in this case, is to move from arithmetic as the base to algebra as the target. As depicted in Figure 2, the model is capable of linking arithmetic to algebra.

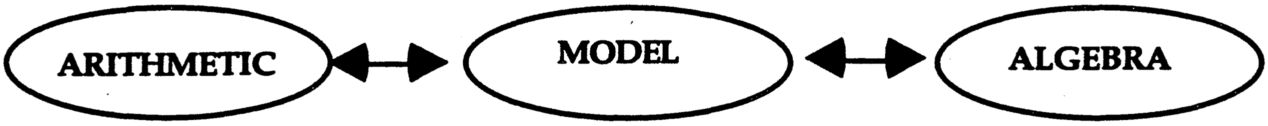

Figure 2. A model as a bridging analogy between arithmetic and algebra

In the light of this realization, the worksheets for introducing students to equivalent expressions with the aid of the objects-and-containers model have been revised so that the first activities make use of sets of objects to represent *numbers (Node A)* and to clarify the arithmetic processes of forming *a sum (B), a product (C), a multiple of a sum, (D) and equivalent forms of the same number (E).* Revision of such arithmetic concepts, the *nodes* in Gentner's (1982) terms, should precede the use of the model for algebra (with arbitrary numbers in containers).

Thus, using sets of objects, the students could be asked to build a set of 3 objects *(Node A)* and a set of 5 objects and use these to show the sum of 3 and 5 *(B)*. Then they could build two more lots of 3 + 5 to show 3(3 + 5) or three lots of 3 + 5 *(D)*. The objects could then be re-arranged to show equivalent numbers such as 3 x 3 + 3 x 5 or 13 + 11 *(E)*. When the notion of a variable number *(Node F)* is introduced to be represented by an arbitrary number of objects (y) placed in a container, the following sequence of algebraic expressions could be modelled:

y*(Node F)* ; y + 5 *(Node G)* ; 3(y + 5) *(Node I)* ; 3y + 15 *(Node J)* ; y + 3 + 2(y + 6) *(Node J)* .

In a similar way, the area model used in School C has been revised by supplying sets of cut-out shapes which enable the students to explore factorization and expansion entirely in the arithmetic system before they progress to use the same principles in the algebraic world. Thus they manipulate composite rectangles to show such relationships as $2 \times 3 + 2 \times 4 = 2(3+4)$ and $(3+4)(2+3) = 3 \times 2 + 3^2 + 4 \times 2 + 4 \times 3$ before $P \times Q + P \times R = P(Q+R)$ and $(R+P)(Q+R) = RQ + R^2 + PQ + PR$.

The revised versions of worksheets will be used in 1995.

Research Plans for 1995

During May 1995 data will be collected from 20 classes and further data may come from another two classes a little later. The researcher is to supply the worksheets for the 600 or more students involved. All classes will complete a pretest and, after four teaching intervention periods, a posttest. The diversity of teaching approaches under study will cover Arithmetic, Symbolic, and Concrete styles. For five pairs of classes an experimental design will be used, having the same teacher use different styles in each class. The other 12 classes will be allocated particular styles or mixtures of styles. The area model and the objects-and-containers model, discussed above, will be used by some Year 8 and some Year 7 classes respectively.

A questionnaire to identify the students' preferred thinking/learning styles will be completed soon after. This will be based on the Ned Hermann (1988) approach and should provide much more useful data than that obtained using a digit-span measure.

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