The Cognitive Structure of Problems Solved Easily and Not So Easily by Pre-Service Teachers

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The purpose of this study was to compare the structure of mathematical word problems which first year pre-service teachers were able to solve easily, with those that they found difficult. The aim was to identify a hierarchy of problem types to which student teachers should be exposed if they are to develop their mathematical problem solving skills. Task analysis maps of twelve word problems given to fifty-eight student teachers suggested a progression in the required level of processing, the nature of the information stated in the problem, the extent of demand on the working memory, and the type of concepts and processes required.

This study was part of a larger project which arose from long-time concern about the inadequacy of mathematical skills of many students entering primary and early childhood education courses at the Launceston campus of the University of Tasmania. A range of tests has indicated that many of these students perform below acceptable levels. This lack of sufficient mathematical skills continues to be a problem despite the importance of primary and early childhood teachers being at least functionally competent at mathematics (Department of Employment, Education and Training, Australia, 1989). The problem is not confined only to a local context. It exists on a much wider basis and is further compounded by the fact that many pre-service primary and early childhood students have poor attitudes towards mathematics (Aiken, 1976; Clark-Meeks, Quisenberry and Mouw, 1982). A potential consequence of not addressing the problem is that pre-service teachers with low skills and high mathematics anxiety become "teachers who lack confidence in mathematics and, as a result, are reluctant and ineffective mathematics teachers who find it difficult to adapt new curricula into their classrooms .... another generation of teachers who are deficient in mathematics is created, thus perpetuating the cycle" (Bitter, 1987, pp.106-7).

In an analysis of tests of basic mathematics concepts given to 260 pre-service primary and early-childhood teachers over a six-year period (Taplin, in press), students performed satisfactorily on the basic operations, simple expressions with fractions, discount, and simple exponents. There were several topic areas identified in which their performances were unsatisfactory. These included multiplication and division of decimals, fraction concepts including equivalents and operations, volume, area, ratio, the relationship between fractions, decimals and percentages, and the application of geometric principles. Performance was particularly poor when the questions required knowledge to be applied in a problem- solving context. This poor performance on problem-solving questions is alarming, considering the fact that organisations such as the Australian Education Council (1991) and the National Council of Teachers of Mathematics (NCTM, 1980 and 1989) have recommended that the mathematics curriculum should be organized around problem solving. If teachers are not competent and confident at solving problems themselves, it is unlikely that they will use the approach effectively in their future classrooms.

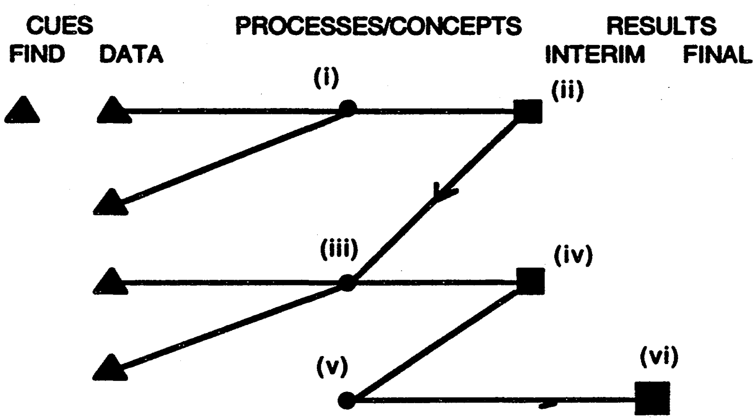
Consequently, it is important to find out more about their strengths and weaknesses in the problem solving domain in order to be able to improve their performances. This need has prompted further analysis of word problems on which the students performed poorly and those on which they performed well. The first part of the analysis was to look at the structure of the tasks themselves, and the next stage will be to explore the cognitive structures of the students' responses to these tasks. The study reported here examined the specific question, are there any differences in the cognitive structures of word problems on which student teachers perform poorly and those on which they perform well?

Methodology

Fifty-eight first and second year primary and early childhood Bachelor of Education students were given a number of word problems, of which twelve typical examples have been selected for discussion. These twelve problems were chosen because they embedded the mathematical knowledge identified by Taplin (in press) as being deficient in comparable student groups. The problems ranged from very easy, which 95 per cent of the students had correct, to very difficult, which were solved by only 12 per cent of the students. The structures of these problems were analysed using a mapping procedure based on that developed by Chick, Watson and Collis (1988) and modified by Collis and Watson (1991), to give a visual representation of the tasks. A discussion of these mapping techniques and the way in which they have been adapted for the purpose of this study is presented in Taplin (1994).

Figure 1 gives an indication of how a map can be interpreted. The student selects data to which a process (i) is applied. This leads to an interim result (ii). More data are needed for the next piece of processing to occur (iii). This leads to a further interim result (iv). The student then checks this result against the initial cues (v) and is satisfied with a correct solution (vi).

Figure 1: Sample task analysis map



Analysis

In analysing the structure of the tasks, the following features have been taken into account:

(i) the level of processing required to solve the task (Collis and Watson, 1991),

(ii) the extent to which the solution can be obtained by using just the cues stated in the problem without having to involve other concepts and processes from the student's previous experience (Collis and Watson, 1991),

(iii) the complexity of the cues stated in the problem and the other implicit concepts and processes required,

(iv) the extent of the demand on working memory (Stillman, 1994).

To investigate the level of processing required to solve the task, the SOLO Taxonomy (Biggs and Collis, 1982, 1991) was used. The problems were assigned, by two independent experts, into those which could be solved using ikonic mode, those which required concrete-symbolic thinking, and those which combined the two modes. They were further classified into three of the response levels defined by the SOLO Taxonomy (Biggs and Collis, 1982, 1991): unistructural, multistructural and relational. The SOLO Taxonomy was also used to explore the complexity of the stated cues and other implicit concepts and processes. Table 1 indicates the definitions which were used for this purpose. The demand on working memory was measured by the number of interim results which were required before the final result could be obtained, and the necessity for more than one process being required to compute a single response.

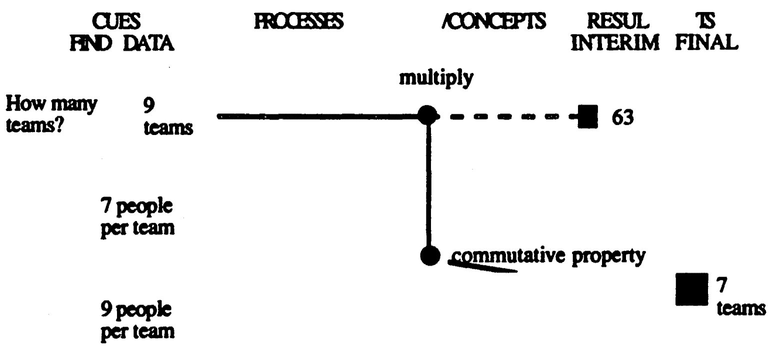
Table 1: Classification of cues and concepts/processes

SOLO Classification	Definition	Example of cue	Example of concept/process brought in from previous experience
Unistructural	can be applied directly with no need for further interpretation	7 teams 9 people	n/a
Multistructural	further interpretation / understanding required before application	area = $96cm^2$	Find factors of 96 to get possible lengths and widths
Relational	integration with other concepts required	volume of box 1=100 dimensions of box 2 = 2x dimensions of box 1	Use algebra to derive volume formula for box 2 from volume formula for box 1

Figures 2-4 show examples of task analysis maps of problems which the students found progressively difficult. It should be noted that the solution paths represented here are not necessarily the most efficient solutions which would be followed by experienced mathematicians. Rather, they represent solutions which are achievable within the limitations of the knowledge and experience typical of the target group of student teachers, which in some cases means the lowest level of processing required to find a successful solution. The problem shown in Figure 2 is a typical example of a problem which the students found easy (95 per cent correct).

Figure 2: Task analysis map of easy problem

How many 9 person teams can be made from nine 7-person teams?



The task analysis map in Figure 2 indicates that, in terms of the SOLO Taxonomy (Biggs and Collis, 1982), this problem can be solved using concrete-symbolic strategies with ikonic support, and is a simple multistructural one because it calls for the integration of two processes. The cues have been classified as unistructural (see Table 1). Some students might solve the problem by including the intermediate response of 63, indicated by the broken lines in Figure 2. Other students would omit this interim response and apply the commutative law directly. The question may require knowledge of one arithmetical rule, the commutative law, which is not provided in the problem. It is, however, possible to solve the problem without knowledge of this rule, by applying division as the inverse of multiplication.

The task analysis map in Figure 3 shows a typical correct response on a problem which approximately one half of the students (45 per cent) could answer correctly. This is a relational problem in the second cycle of the concrete symbolic mode (Campbell, Watson and Collis, 1992), because it requires the integration of a sequence of separate pieces of information about area and perimeter . It requires three interim results in order to obtain the solution. This problem provides three cues which are all consistent with the definition of multistructural in Table 1. There are three multistructural processes which need to be applied before the solution can be obtained: the respective formulae for the area and the perimeter of a rectangle, and the conversion of millimetres to centimetres. There are also two instances where two processes are required to produce a single interim response. An example of this is the need to apply the area formula in order to recognise the need to find factors of 96 (i).

Figure 3: Task analysis map of difficult problem

A rectangle has an area of 96 square cm and a perimeter of 560mm.
What are its length and width?

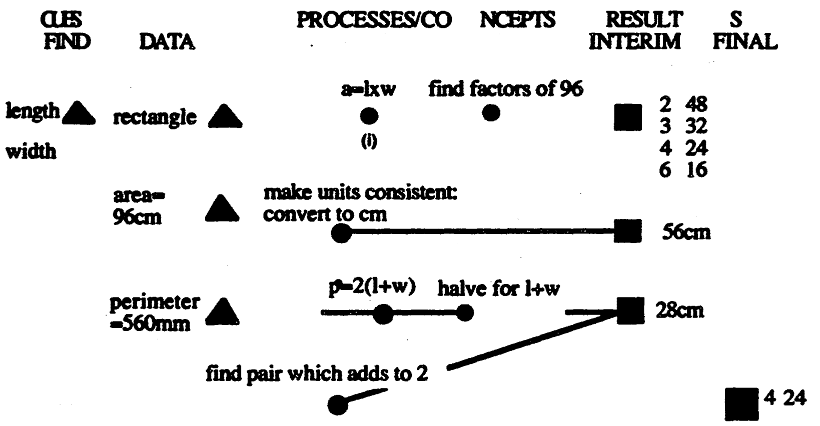
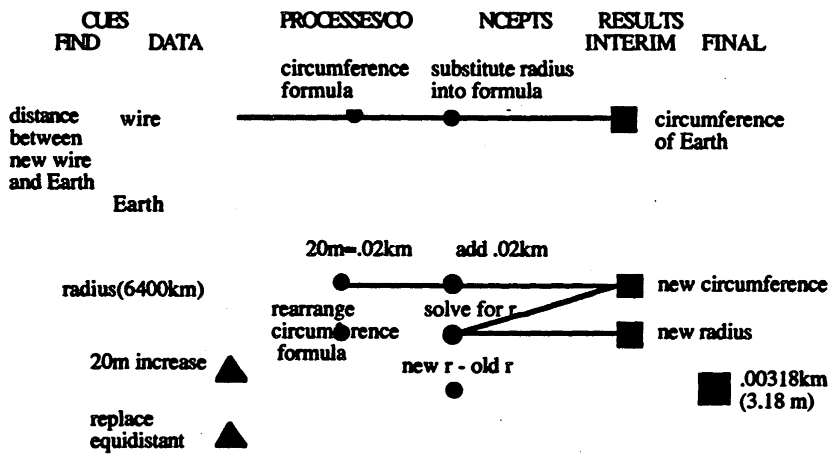


Figure 4 shows an example of a map which the students found to be very difficult, with only 15 per cent being able to obtain a correct solution. This is also a relational problem in the second cycle of the concrete-symbolic mode, which involves "an integration of the intermediate responses" (Collis and Watson, 1991, p.76). It is more complex than the relational problem in Figure 2 because it involves operations on decimals and pi, whereas the former problem is concerned with whole numbers. It requires three interim results before the solution can be found. On three occasions, it is necessary to combine two processes to produce an interim response. These are: knowledge of the circumference formula and ability to substitute for the radius to be able to calculate the circumference, conversion of 20 metres to kilometres, and ability to rearrange the circumference formula and find the radius. There are five cues stated in the problem. The first two of these can be classified as multistructural and the latter three as relational. The problem implies the understanding of four concepts from the student's previous experience: that the length of the wire represents the circumference of a circle (multistructural); calculation of the circumference of a circle, given the radius (multistructural); recognition of the inconsistent units and conversion of 20 metres to kilometres (relational); and transformation of the circumference formula to calculate the radius of a circle (relational).

Figure 4: Task analysis map of very difficult problem

Suppose a wire is stretched tightly around the Earth. (The radius of the Earth is approximately 6400 km.) If the wire is cut, its length is increased by 20m, and it is placed back around the Earth so that it is the same distance from the Earth at every point, could you walk under the wire? How far would the new wire be above the Earth?



A summary of the analyses of the twelve problems is shown in Table 2. The problems have been divided into three categories: those on which more than one-third of the students in the sample were correct (easy), those on which between one-third and two-thirds succeeded (medium), and those on which fewer than one-third of the students were able to obtain correct solutions (difficult). Problems in each of these categories were compared to identify common properties relating to the criteria listed above.

Level of processing required to solve the problem

It is clear that the problems on which the majority of students were successful are those which can be solved in the ikonic (IK) mode or concrete-symbolic mode with ikonic support (CS-IK), using simple multistructural responses. Fewer students were able to solve the two multistructural problems in the concrete-symbolic (CS) mode, with 38 and 47 per cent respectively being incorrect. Four problems can be classified as level 2 relational in the concrete-symbolic mode. These were not solved by many students, with 55 per cent unable to solve the easiest and 88 per cent the most difficult.

Extent to which solution can be obtained by using just the cues stated in the problem without having to involve other concepts and processes from the student's previous experience

Of the easiest group of problems, there is only one which requires the use of implicit concepts from the student's previous experience. This is knowledge of the commutative law, classified as a multistructural concept (represented as M in Table 2). Three of the four medium problems require the use of at least one implicit concept. The set of difficult problems requires three or more implicit concepts to be used. In each case, at least half of these concepts are relational (represented as R in Table 2). There do not seem to be any consistent differences in the numbers of cues given in the easy, medium or difficult problems, but there is a suggestion that they increase in complexity. Cues in the easier problems can be classified as predominantly unistructural (represented as U in Table 2), whereas those in the most difficult are multistructural or relational.

Extent of the demand on working memory

For this analysis, the demand on working memory was determined by the number of interim results which are required before the solution can be reached, and the number of times two or more processes are combined to produce an interim result. The easy questions do not require very much information to be held in working memory. In most cases there are no interim results needed. The medium problems impose a slightly greater demand, with three of them requiring one interim result . The difficult problems can be regarded as multi-step, necessitating three or more interim results. Three of the medium problems call for two processes to be combined for one interim result. Two of the difficult problems require this level of processing on more than one occasion.

Table 2: Summary of task analyses

Problem*	% of students incorrect	Number of interim results required	Required level of processing (SOLO Taxonomy)	Number and types of cues given	Concepts/ processes required but not provided	Number of times 2 processes combined to produce one interim result
>30% of Ss Correct						
1. Commutative law	5	0 or 1**	CS-IK/Multistructural	3 U	1 M	0
2. Boys in clubs	13	2	IK/Multistructural	4 U	0	0
3. Shaded fraction	19	0	CS-IK/Multistructural	3 U	0	0
4. Fraction recognition	20	0	CS-IK/Multistructural	2 U	0	0
5. Scale	23	0	CS-IK/Multistructural	4 U	0	0
Between 30% & 60% of Ss Correct						
6. Ratio	31	1 or 3**	CS-IK/Multistructural	4 UUUM	1 M	0
7. Chocolate	38	0	CS/Multistructural	3 UUM	1 M	1
8. Percentage discount	47	1	CS/Multistructural 2	2 UM	1 M	1
9. Area/perimeter	55	1	CS/Relational 2	3MMM	3 MMM	2
<30% of Ss Correct						
10. Volume of cube	71	3	CS-IK/Relational 2	2 M R	3 MRR	0
11. Tank	85	4	CS/Relational2	3 MMR	3 RMR	2
12. Wire	88	3	CS/Relational2	5 MMRRR	4 MMRR	3

* A full description of these problems is available from the author.

**solvable in two different ways

Discussion and Implications

The mapping procedure used in this analysis provides valuable insights into the cognitive structure of the problem-solving tasks on which pre-service teachers performed well and those on which they performed poorly. The findings suggest that there were differences in the cognitive structures of problems which pre-service teachers found easy and those which they found difficult. They perfomed best on unistructural questions which could be supported by ikonic thinking, where all necessary information was given in the problem in unistructural form, and only one step was required to reach the solution. They performed worst on level 2 relational questions where the cues were more complex, it was necessary to employ several multistructural or relational concepts from their previous experiences, several interim results were required, and more than one process was needed to calculate an interim result. There is support here for the argument presented by Tarmizi and Sweller (1988), that students are more successful in solving problems which do not impose a heavy cognitive load by requiring them to integrate multiple sources of information. While these results are not surprising, there are some clear implications for the kinds of problem-solving experiences which should be incorporated into the pre-service mathematics education programme if we are to prepare teachers who are competent problem solvers. The outcomes suggest a hierarchy of problem types which student teachers should experience. In particular, they need exposure to problems which do not provide all of the necessary information, so they can learn to identify and include concepts from their previous experiences. They need to solve problems which require them to interpret and integrate the information stated in the problem and to do the same with the implicit concepts and processes. They also need multi-step word problems, so they learn how to sort out the data and not give up after the first step. If teachers are not encouraged to solve this type of problem, the learning experiences they offer their children will remain restricted to the lower level problems with which the teachers feel comfortable. If they are presented with tasks which are designed to develop their competence, it is likely that confidence will also develop.

The analysis presented in this paper has contributed towards understanding the types of problem-solving tasks which should be included in the teacher education programme. It is still necessary to explore how students can develop the strategies for solving this type of problem. To contribute to this knowledge, the next stage of the research will be to explore the cognitive structure of the pre-service teachers' responses to various types of problems.

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