The Role of Physical and Cybernetic Phenomena in Building Intimacy with Mathematical Representations

James J. Kaput
University of Massachusetts-Dartmouth

We examine the educational possibilities afforded by new connections between physical and simulation-based data to build intimacy with function representations. Historically, technology was first used to facilitate actions within notations, then to link them, eventually bi-directionally. Yet recent data strongly indicate that students' difficulties with interpreting and productively using mathematical notations continue. We suggest that students need phenomena expressive linked to the notations, and secondly, that students themselves should be immersed in generating such phenomena.

General Introduction

Simulations and computer-based models are proving to be the most powerful resources for the advancement and application of mathematics and science since the origins of mathematical modeling in the Renaissance which led to the development of algebra, calculus, and the historical explosion of knowledge and technology that followed. The move from static models built in inert media using traditional notations to dynamic models and associated visualization and analytic tools built in interactive media are changing profoundly not only the nature of mathematics and science but the nature of inquiry in these disciplines (Glass & Mackey, 1988; Haken, 1981). These changes in inquiry involve both a change in the kinds of phenomena that must be considered, and a change in the nature of argumentation and acceptable evidence (Holland, 1995). Furthermore, these cybernetic tools are finally beginning to appear in education. While their appearance is recent, their educational presence will accelerate into the next century as cybernetic experience begins to infiltrate all aspects of human activity, and as the move from static, inert models to dynamic, interactive ones is followed by moves to ever more immersive simulations and eventually to virtual reality (Heim, 1993). This evolution in modes of experience will be accompanied by an evolution from classical mathematics and science to nonlinear science and mathematics, where iterative, dynamical models will extend the range of phenomena subject to systematic modeling and investigation (Pagels, 1988).

However, to introduce these powerful tools and methods, fully formed in the ways that they are used by mature scientists, mathematicians, engineers, or other professionals who apply them is akin to introducing the classical tools of mathematics and science full-blown, to novices (AAAS, 1989; Richards, Barowy, & Levin, 1991). We know that this does not work, and the history of mathematics and science education is, in part, the story of dealing with this fundamental fact. We must now come to terms with the new complications associated with these new tools, especially for younger students, for whom non-cybernetic experience is simultaneously critically important yet under exploited in school learning. At this point, we need basic principles that would help structure productive interactions between physical experience and cybernetic experience in mathematics and science across all age levels (White, 1993).

A Look Back Over Recent Uses of Multiple Representation Computer Systems

Over the past ten years, computer technology has made possible and popular the physical linking of mathematical representations, most especially, graphs, tables and formulas, usually in the context of teaching and learning algebra and calculus. In recent years, the linking has become bi-directional, and much student activity and exploration becomes possible exploiting the interactive nature of the technology. Figure 1 has become a commonplace in descriptions of curricula, texts and learning objectives (Confrey, 1992; Kaput, 1986, Kaput, 1989; Lesh, et al., 1987). You will note that in Figure 1 the "symbolic formulas" vertex is exaggerated, to reflect its dominance in practice. Indeed, character string notations are "more equal" than any of the others, as is evident, for example, if one looks at how functions are input on most devices and how functions are named or identified in activities - as algebraic formulas.

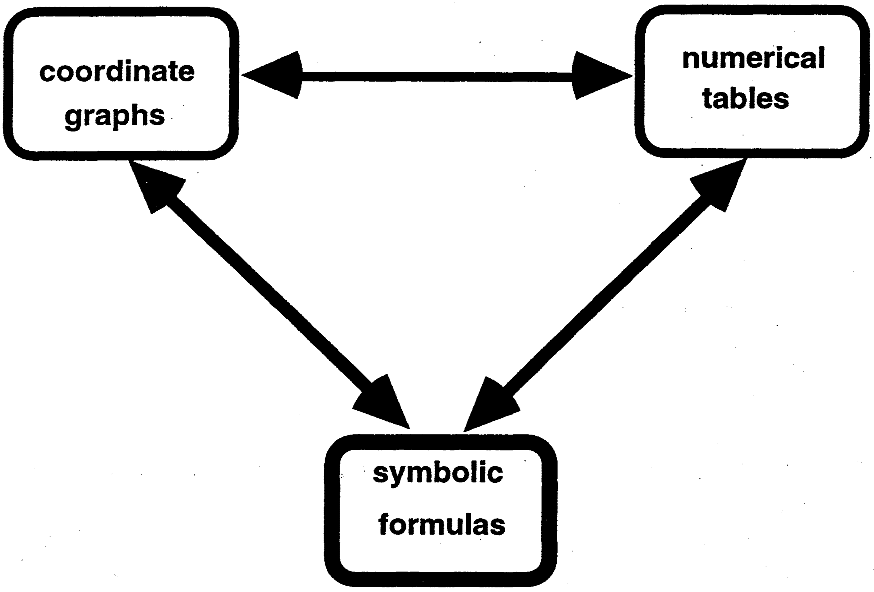

Figure 1: The "Big Three"

However, it is becoming increasingly apparent that student difficulties in learning and using functions have not disappeared (Eisenberg & Dreyfus, 1994; Goldenberg, 1988; Sfard, 1995; Teles, 1989; Thompson, 1994b; Vinner & Dreyfus, 1989; Yerushalmy, 1991). Indeed, the detailed analysis by Schoenfeld, Smith & Arcavi (1994) exhibits the extraordinarily fragile nature of understandings of linear functions when those functions are studied strictly in terms of graphs, formulas and tables apart from referential anchors in students' experience. Whether or not the physical (computational) links become conceptual links, understandings seem inadequate and incomplete. One may ask why this is the case. Our reply heeds the words of Anna Sfard (PME-NA 1994 Plenary Address):

"The emperor is *only* clothes!"

Representations need to represent *something* (other than each other). They need anchoring in student's physical, imagistic, emotional experience. But this, too, is a widely held maxim. "Connections" play a central role in the NCTM Curriculum and Evaluation Standards (NCTM, 1989) and in most reform rhetoric. The connections to the wider world are depicted in Figure 2, where the text descriptions of situations are typically offered in text, as "word problems," and where the connections are in the minds of the problem writers and, presumably at some point, in the mind of the student. Hence the lighter arrows.

As is indicated in Figure 2, the historical approach attempts to link the mathematical notations to the wider world of phenomena and situations in a weak and abstract manner that depends on a *target* student ability to interpret *textual* descriptions of phenomena and situations that are not themselves experienced in any direct form. Further, student ability to link conceptually the textual descriptions of phenomena and situations is typically ill-formed and far from robust. The mathematical side of the curriculum seeks to use the phenomena and situations to help learn mathematics, while the science side of the curriculum seeks to use the mathematics to deepen understanding of the phenomena and situations. Indeed, science instruction, much more frequently than mathematics instruction, attempts to provide some

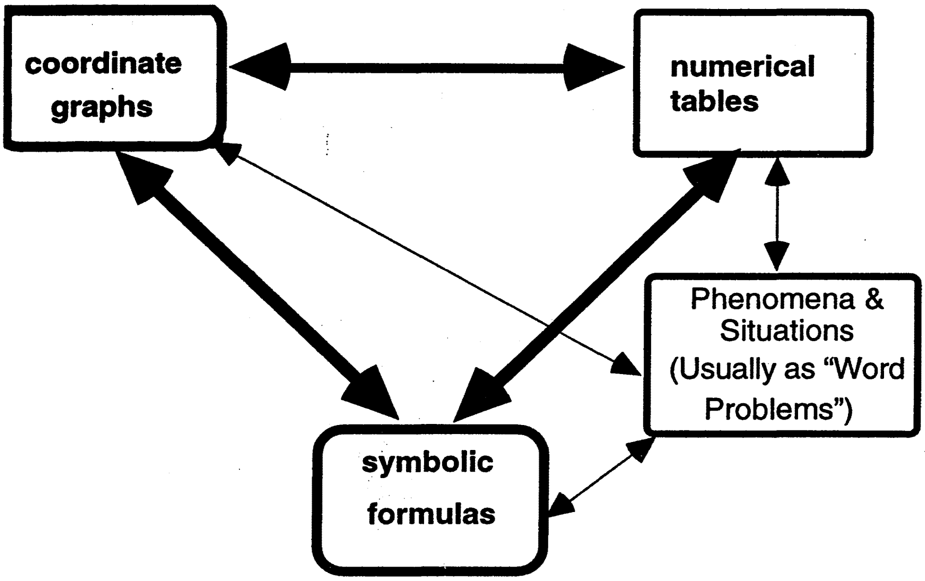

Figure 2: Linking the Big Three to Phenomena & Situations - Weakly

experiential version of the phenomena as part of the instructional context. *Nontheless, we suggest that the mutual constitution of meaning for not only the notations and links among them, but for the phenomena and situations that they may be used to model, is insufficiently rooted in authentic student experience.* Put simply, from the perspective of the students, the phenomena and situations either are missing altogether or are inadequately linked to the mathematics that is to be involved. The instructional strategy offered in this paper is to offer stronger and more direct connections to phenomena and situations, as reflected in Figure 3.

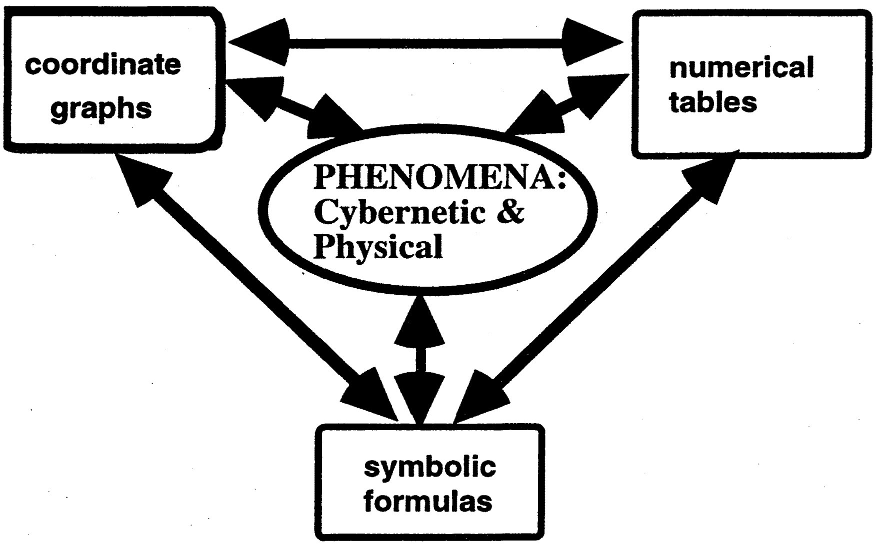

Figure 3: Stronger Links to Directly Experienced Phenomena

The kinds of phenomena that may be appropriate vary widely depending on the learning and teaching objectives. Ideally, experience of these phenomena and situations should both embody a measure of generality as well as deliberately tap into the expressive, imagistic, kinesthetic, linguistic, personal, social and other resources that students bring to the learning situation. Since our instructional target is the Mathematics of Change, as described below we begin with motion phenomena, which include many of the desired characteristics, including a readily transferable descriptive vocabulary: words and phrases such as "speeding up," "slower," "catching up," "stopping," "turning around," "back and forth," "going at a constant rate," etc., are widely applicable to describe change phenomena whether or not they involve physical motion. That is, this language applies to many different kinds of quantities other than velocity, position or acceleration. Furthermore, they engage kinesthetic resources based in the first person experience of motion as well as imagistic thinking based in the varieties of graphical representations of motion. Finally, through an appropriate choice of activities, students can become engaged in expressive activities and even aesthetic concerns usually associated with dance and collective motion.

Relating Cybernetic Vs Kinesthetic Experience: Illustrations Involving the Mathematics of Motion

The forms of interaction between computer-based and non-computer-based experience can take many forms depending on the domain and the sophistication or age of the learner. We have been concentrating on the mathematization of motion, an intellectual effort that began with the Scholastics (Clagget, 1968; Kaput, 1994), continued through Galileo and Newton (Edwards, 1979), and is re-emerging as an item of interest in the context of nonlinear phenomena modelable via dynamical systems (Nemirovsky, 1995; Prigogine, & Stengers, 1984; Sandefur, 1990, 1993). Of considerable importance is the fact that our efforts to include this activity in the earlier grades requires us to engage students in characterizing their motion via graphical, narrative, and kinesthetic methods rather than using algebra - as was the case historically well before the time of Newton, because algebraic tools were unavailable (Kaput, 1994). While a large body of research exists regarding student mathematization of motion, it mainly focuses on high school and college age students, and generally deals with "regular" motion that is describable algebraically (McDermott, 1984; Thornton, 1992). Indeed, student modeling of constant-acceleration and harmonic motion is usually a goal of this work. However, we are interested in modeling motion (1) beginning in the early grades, and (2) beginning with the "irregular" motion of children's own bodies and objects controlled by them.

The move to younger children heightens the importance of concepts of rate, another well-studied area (Piaget, 1970, 1976; Piaget & Inhelder, 1974; Thompson, 1994). Indeed, work with speed and velocity provides an excellent arena for the learning of rate ideas, and we are especially interested in tracking the growth of concepts of rate where, unlike the standard approach, students are confronted with variable rates early and develop a sense of constant rates as a means of handling the complexity of these variable rates - via the development of a sense of average speed, or mean value. It may be worth pointing out that mean value is, mathematically, an important concept upon which much theoretical structure depends as is evident from an examination of the proofs of most any university calculus text (e.g., Fleming & Kaput, 1979).

Relating Kinesthetic and Cybernetic Motion: The SimCalc Project Approach

While we often use the phrase "motion simulation" to refer to our computer-based motions, it is important to realize that screen motion of objects amounts to visually real motion to which all the basic perceptual and analytic resources can be applied. They yield visually experienced motion, whether they are subject to first person or third person control (as for example, the difference between a simulated car that is being "driven" in real simulation time from within with a windshield view, etc., and a car observed from the outside whose motion is specified by giving its velocity graph, for example). Nonetheless, they do not yield kinesthetically experienced motion, although it is also possible to create a motion by dragging an object via mouse-control on the screen, which involves a quite direct connection between hand-motion and simulation-motion (Stroup, 1993, 1996).

The physical world provides *kinesthetically rich* experiences of motion, but *quantitatively poor* experiences - where time and position are difficult to measure and use to quantify velocity. This presents an especially difficult challenge for young students who have only the vaguest idea of what one might measure and why. (One need only recall the highly stylized and tightly choreographed "experiments" in college physics laboratories aimed at enabling students to measure acceleration due to gravity.) As a contrast, an appropriate computer simulation can provide quantitatively rich experiences, but, obviously, kinesthetically impoverished ones. For example, one can engage students in graphically controlling and observing motion of a computer-based elevator in a building, where each floor of the building is clearly numbered, where hot-linked position and velocity graphs are labeled, time is visually marked by a sweeping time cursor, and all relevant quantities are directly and easily adjustable. Indeed, one can enable the students to control multiple elevators and compare motions one against the other as they are repeated and/or changed.

The SimCalc Project (Kaput, 1995) has produced a whole series of motion simulations involving objects or characters moving in various simulated "worlds" that students can control in various ways - swimmers in multilane pools, dancers on a ballroom floor, people and animals moving across various scenes (e. g., duckies on a pond, people walking along a path), cars and trains that can be "driven," and even a schematic "dots world," where the student controls the movement of dots across a simple scrollable, zoomable, numerically labeled grid. However, unless the screen-object motions produced by the students can be linked to their personal experience of motion in more than an abstract way, their motion experience is only visual and disconnected, much as in a video game (Kaput, 1994). It does not support connections with knowledge about their movement in space, or the movement of other physical objects. We are studying two ways of building such connections, varying in their level of student interactive intimacy with the motion data:

(1) Connections based on alternating between physical and cybernetic motion, where students engage in activity structures in one realm designed to support learning in the other; and
(2) Connections based on integrating simulation-based and physically-based data collected in real time within the same computer system.

<u>Examples of 1</u>: Connections of the first type can take the following form, beginning with physical motion, partially inspired by the pioneering work of Mary Barnes (Barnes, 1993). Versions of these activities have been piloted with 9 and 10 year-olds (Kaput, 1995) and a larger scale study involving more than 50 students in grades 3-5 is underway as of this writing. Children walk and run along a numerically labeled straight path with speeds characterized qualitatively as "slow," "medium," and "fast" where measurements take the form of timing "trips" along pre-specified distances, noting distances traveled over given time intervals, comparing different students' versions of slow, medium and fast, etc. This limited 3-valued version of velocity matches children's limited ability to discern and discuss velocity differences. The next activities are on-line with, say, an Elevator (vertical) or "Walking World" (horizontal) simulation, that are likewise initially restricted to 3 speed values, but where the values are explicitly given as quantities, say 1 floor/sec for "slow," 2 floors/ sec as "medium," and 4 floors/sec as "fast." These values are represented graphically on a velocity graph as horizontal straight line segments whose student-adjustable length determines the duration of the motion as depicted in Figure 4. In this situation, the student drags a choice of S, M, or F segment onto the velocity graph and stretches it to the desired width (duration). This can be done for two or more elevators to compare motions associated with different graphs.

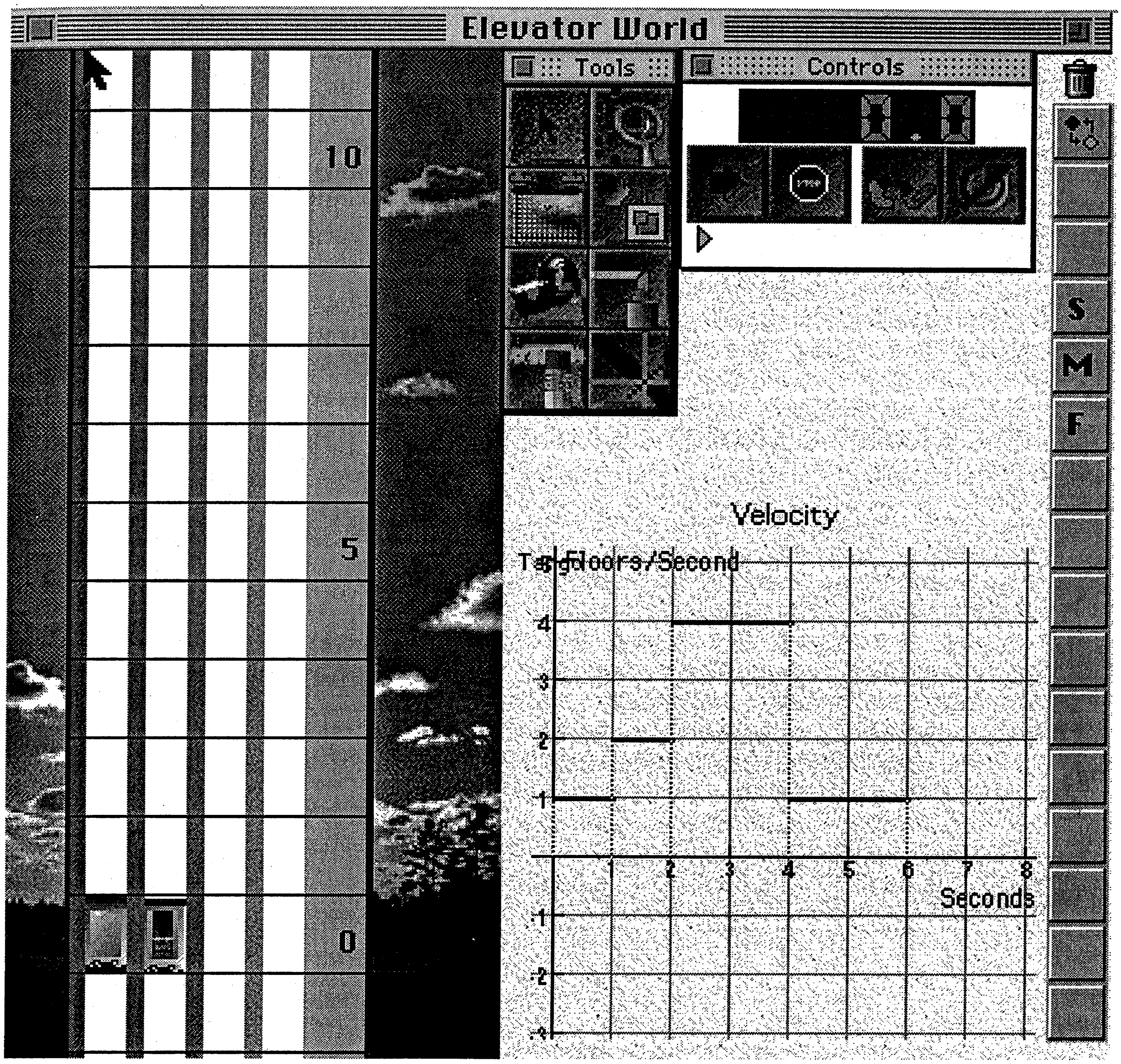

Figure 4: Elevator World

Students examine how far elevators travel over different time intervals at these different speeds. In the simulation, the experience is strongly quantitative, with much computation associated with testing conjectures. ("Snap-to-grid" is turned on so that all time and velocity values are whole numbers, reducing computational complexity.) Gradually, they are introduced to variable speed elevators, mixing slow, medium and fast speeds, and confront such questions as whether one could get to a given floor with a constant medium speed at the same time as an elevator that first goes slow, then fast. This exercise in determining mean values amounts to adjusting the height of the velocity graph (which some students begin to relate to the area under the horizontal velocity graph), but it is devoid of physical meaning until the students return to the physical activity with the same activity-structure - where they now move in pairs. Here their kinesthetic sense of "catching up," for instance, on a slow-then-fast trip simultaneous (side-by-side) with a partner's medium speed trip, often confronts their tendency to race. This type of situation, where maintaining a constant speed is problematic, raises questions of what constant speed means and yields a deeper, more connected meaning for average and constant rates. Later, students return to the computers to deal with more complex versions of the mean-value activity than could be handled physically, where the visually editable graphs make this possible. To introduce negative velocities, students again return to the physical realm, where they act out and measure times for various types of "round-trips," for example, trips that go slow in one direction and fast in the other. When they return to the computer, where restrictions to positive velocity values are removed, they confront the question of what kind of velocity graph will make the elevator go downward. Once again, the physical experience takes more precise quantitative form as they compute positive and negative areas (either by multiplying length dimensions of the rectangles involved, or by counting grid-squares).

Examples of 2: Physical/cybernetic connections of the 2nd type are based on newly available functionality that builds on the traditional computer-based laboratory ability to import and graph quantitative data from measuring instruments on a real-time basis.

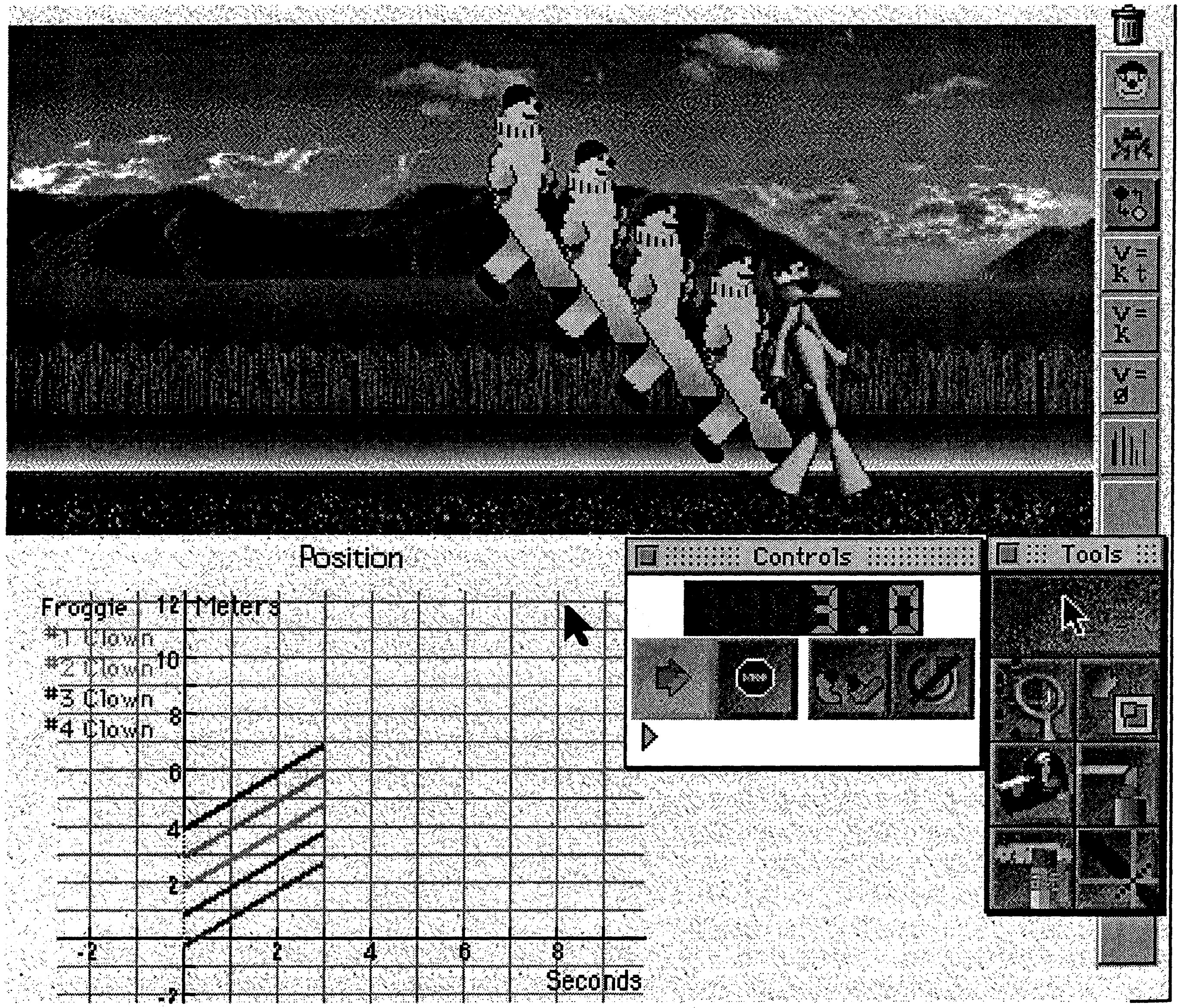

Figure 5: Leading Your Own Clown Parade

Traditional Micro Computer Based Laboratory (MBL) tools enable one to input physically based data into the computer and display it in various ways (Thornton, 1992; Tinker & Thornton, 1994), including and especially motion-based data. We can now not only graph the data describing, say, a student's physical motion, but we can attach that data to a character or object in a simulation and replay the motion so that it can be studied for subtlety that may not be evident in its initial transient enactment. But more importantly, this imported data can be mathematically transformed and compared with the original, and students can interact with either the original or transformed data in a variety of ways. For example, they may try to create physically the motion of the transformed data (and then compare their attempts with the "original" transformed data), they might try to approximate the original data with that generated by an algebraic formula, they might try to create physically a motion that matches the algebraic data, they might compare their motion with some particularly interesting aspect of motion of a classmate, they might try graphically to find a mean-value for a physically generated motion, and so on. In all these cases, they can examine all attempts both from multiple perspectives: coordinate graphical, algebraic, numerical, and via actual screen motions. This kind of intimate interaction between the student's physical experience and mathematical experience, while sharing some aspects of what Papert (1993) refers to as "syntonic" experience, is both new and a harbinger of a whole new style of instruction that mixes "first person action" and "third person" observation and adjustment of mathematical objects. One can regard it as yet another step in the broad evolution of integrating cybernetic experience into human activity (Heim, 1993). For example, in Figure 5, the "froggie" could enact your imported physical motion, which is used as the template for a collection of clowns whose motion is graphically edited so that they follow you in a parade (or, alternatively, move in some interesting counterpoint to your motion).

Our aims are broader than simply understanding motion. Ultimately, we want students to exploit their understandings in other settings, with quantities other than velocity, position, etc. In the terms of Figure 2, we want students to make the wider connections to their world that were among the traditional objectives of mathematics instruction. Accounts of students achieving these connections are in preparation.

References

American Association for the Advancement in Science. (1989). *Science for all Americans: A Project 2061 report on literacy goals in science, mathematics, and technology.* Washington, DC.

Ainley, J. (1995) Re-viewing graphing: Traditional and intuitive approaches. *For the Learning of Mathematics, 15*(2) June, 10-16.

Barnes, M. (1993) *Investigating change: An introduction to calculus for Australian schools.* Carlton, Victoria: Curriculum Corporation.

Clagget, M. (1968). *Nicole Oresme and the medieval geometry of qualities and motions.* Madison, WI: University of Wisconsin Press.

Confrey, Jere (1992). Using computers to promote students' inventions on the function concept. In S. Malcom, L. Roberts & K. Sheingold (Eds.), *This year in school science 1991: Technology for teaching and learning* (pp.141-174). Washington, DC: American Association for the Advancement of Science.

Edwards, C. (1979). *The historical development of the calculus.* New York, NY: Springer-Verlag Inc.

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformation. In E. Dubinsky, A. Schoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education. I,* (Vol. 4, pp. 45-68). Providence, RI: AMS.

Fleming, D., & Kaput, J. (1979). *Calculus & analytic geometry.* New York: Harper & Row.

Goldenberg, E. P. (1988). Mathematics, metaphors and human factors: Mathematical, technical, and pedagogical challenges in the educational use of graphical representation of functions. *Journal of Mathematical Behavior, 7*(2), 135-173.

Glass, L., & Mackey, M. (1988). *From clocks to chaos.* Princeton: Princeton University Press.

Grant, W., & Borovoy, R. (1991). Apple classroom of tomorrow program: Apple Computer, Inc. Cupertino, CA.

Haken, H. (1981). Chaos and order in nature: *Proceeding of the International Symposium on Synergetics.* New York: Springer-Verlag.

Heim, M. (1993) *The metaphisics of virtual reality.* New York: Oxford University Press.

Holland, J. H. (1995). *Hidden order: How adaptation builds complexity.* New York: Addison-Wesley.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence.* New York: Basic Books.

Kaput, J. (1986). Information technology and mathematics: Opening new representational windows. *Journal of Mathematical Behavior, 5,* 187-207.

Kaput, J. (1994). Democratizing access to calculus: New routes to old roots. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving.* Hillsdale, NJ: Erlbaum.

Kaput, J. (1995) SimCalc YR 2 Annual Report to the National Science Foundation. Available from the author and on the WWW SimCalc site.

Kellert, J. (1993). *In the wake of chaos.* Chicago, IL: University of Chicago Press.

Lakoff, G., & Johnson, M. (1980). *The metaphors we live by.* Chicago, IL: The University of Chicago Press.

Lesh, R., Behr, M., & Post, T. (1987). The role of representational translations in proportional reasoning and rational number concepts. In C. Janvier (Ed.), *Problems of representation in mathematics learning and problem solving.* Hillsdale, NJ: Erlbaum.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and the velocity sign. *Journal of Mathematical Behavior, 13,* 389-422.

Pagels, H. (1988) *The dreams of reason: The rise of the sciences of complexity.* [publisher]

Papert, S. (1993). *The children's machine.* New York: Basic Books.

Prigogine, I., & Stengers, I. (1984). *Order out of chaos.* New York: Bantam.

Richards, J., Barowy, W., & Levin, D. (1991). Computer simulations in the science classroom. *Journal of Science Education, 1,* 67-79.

Sandefur, J. (1990). *Discrete dynamical systems: Theory and applications.* Oxford: Clarendon.

Sandefur, J. (1993). *Discrete dynamical modeling.* Oxford: Oxford University Press.

Schoenfeld, A., Smith, J., & Arcavi, A. (1994). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4,). Hillsdale, NJ: Erlbaum.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14,* 15-39.

Stroup, W. (1993) *Kinemouse.* Software. Educational Technology Center, Harvard Graduate School of Education. Cambridge, Massachusetts.

Stroup, W. (1996) Learning the mathematics of motion. [Dissertation] Harvard Graduate School of Education.

Teles, E. J. (1989). Numerical and graphical presentation of functions in precalculus. *Dissertation Abstracts International, 51,* 777A.

Thompson, P. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confey (Eds.), *The development of multiplicative reasoning in the learning of mathematics.* Albany, NY: SUNY Press.

Thompson, P. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, J. Kaput & A. Schoenfeld (Eds.), *Research in collegiate mathematics education* (Vol. 1,). Providence, RI: American Mathematical Society.

Thornton, R. (1992). Enhancing and evaluating students' learning of motion concepts. In A. Tiberghien & H. Mandl (Eds.), *Physics and learning environments* [NATO Science Series]. New York: Springer-Verlag.

Tinker, R., & Thornton, R. (1994). Constructing student knowledge in science. In E. Scanlon & T. O'Shea (Eds.), *New directions in educational technology* [NATO Science Series]. New York: Springer-Verlag.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(5), 356-366.

Yerushalmy, M. (1991). Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning, 7,* 42-57.