Trigonometry: Comparing Ratio and Unit Circle Methods

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Before the 1960s, introductory trigonometry was taught in Victorian schools using the ratio method, where trigonometric functions are defined as ratios of sides of right angled triangles. With the advent of "new maths", the unit circle method was introduced. This study explored differences between the two methods for teaching introductory trigonometry. Eight classes of students were randomly allocated to either teaching method. The ratio method was found to be much more effective, resulting in better performance and retention in trigonometry and algebra, and more favourable attitudes.

Two methods of introducing trigonometry

Since the advent of the "new mathematics", two methods of teaching introductory trigonometry have been used in Victorian schools. For decades, introductory trigonometry had been taught to Year 9 and 10 students (average age 14 and 15) by the ratio method. In this method, the trigonometric functions are defined as the ratios of pairs of sides in a right angled triangle. From the early 1960s, an alternative "modern" method was advocated by some educationalists (Trende, 1962; Willis, 1967) as a more desirable way for students to learn and understand the topic. This approach, known as the unit circle method, defined cosine and sine as the x and y co-ordinates of a point on a unit circle. Both methods are outlined in more detail below. Since 1982, the unit circle method has been the officially preferred method for the Year 9 and 10 mathematics curriculum in Victoria (Secondary Mathematics Committee, 1982; *Geometry Everywhere,* 1991). Currently there are text books promoting each way of teaching trigonometry and a few try to blend the two approaches.

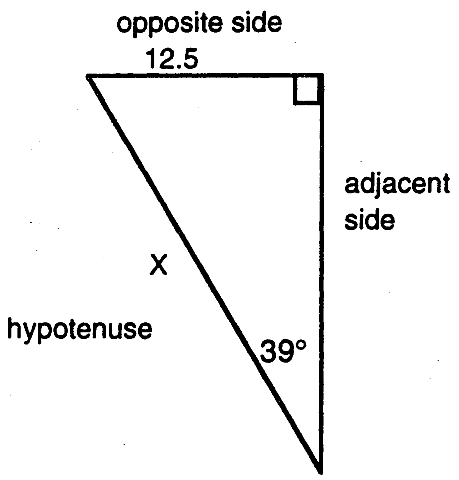
The main purpose of this research was to compare the two methods of teaching basic trigonometry to see which promotes better understanding of the underlying concepts and mastery of skills. In the introductory phase which we are concerned with here, the main skills are to calculate the sides and angles of right angled triangles from known sides and angles. A second goal of the research was to investigate the relationship between success in trigonometry and algebraic understanding. Thirdly, the strategies used by low ability students to achieve success in trigonometry were analysed. Space does not permit the second and third aspects to be reported here, but full details are given by Kendal (1992).

No formal research has been reported which compares the two ways of introducing trigonometry although there has been continuing presentation of anecdotal evidence and opinion. Contributions to the debate on which method should be used and the difficulties experienced by students learning trigonometry include Butler & Lynwood Wren (1970), Ellis (1990), Lloyd (1976), Markel (1982) and McNaughton (1965) discussing the ratio method, Collins (1973), Dooley (1968), Ince and Johnston (1988), Shear (1985), Stokie (1966), Trende (1964) and Willis (1966, 1967) who advocate the unit circle approach and Ellery (1980) and Satty (1976) who favour a combination of the two methods.

When first introduced as part of the "new maths", the unit circle method emphasised the nature of the trigonometry functions as functions taking real numbers to real numbers. One of its virtues was seen to be that no reference to angles or triangles was required, although "the solution of triangles was noted as an interesting and useful outcome" (Dooley, 1968). The trigonometry functions were defined as *functions* of a *real variable,* the length along the circumference from (1,0) to P (see Figure 2). This avoided the undefined notion of angle. Whereas the ratio definitions arise naturally from applications to mensuration and surveying, the unit circle definitions lean more naturally towards applications to periodic phenomena, such as simple harmonic motion. Both of these sources of applications are important. Our question in this study is not which definition should be used - we believe students as they learn more advanced mathematics need both. Our question is which method should be used to introduce trigonometry, given that the principal goal to be achieved at that stage is the solution of right angled triangles.

Description of the teaching methods.

Ratio method. For the ratio method, the trigonometry functions are defined as the ratios of lengths of the sides in right angled triangles. For example, the sine of an angle is defined as the ratio of the length of the "opposite side" to the length of the hypotenuse. Students are often taught to remember the definitions of the ratios using a mnemonic such as SOHCAHTOA (S̲ine =O̲pposite + H̲ypotenuse etc). The named triangle sides are shown in Figure 1, along with one of the hardest calculations that students are expected to do at this early stage. The difficulty of this example arises from the difficulty of solving the algebraic equation.



Find the length of the hypotenuse, x.

$$\sin 39° = \frac{\text{length of opposite}}{\text{length of hypotenuse}}$$

$$\sin 39° = \frac{1.25}{x}$$

$$x = 12.5 \div \sin 39°$$

$$x = 12.5 \div 0.629$$

$$x = 19.873$$

(OR more accurately, x = 19.863 if the value of sin 39° is not truncated before the final step)

Figure 1. Naming the sides of a right-angled triangle and a hard calculation.

Unit circle method. In the unit circle method, the trigonometry functions derive their meaning from the unit circle, as illustrated in Figure 2. Even though this is the definition which applies to all quadrants of the circle, current textbooks only use the first quadrant. To find lengths and angles in right angled triangles, the given triangle is compared with the standard reference triangles (which students are to learn) and then properties of similar triangles are used for calculation. In order to avoid the difficulties of solving algebraic equations, especially with the unknown in the denominator, recent texts adopt a variation of the unit circle method which uses a scale factor comparing the given triangle with the reference triangle. Figure 3 illustrates the procedures involved in a case where it is difficult to visualise the scale factor.

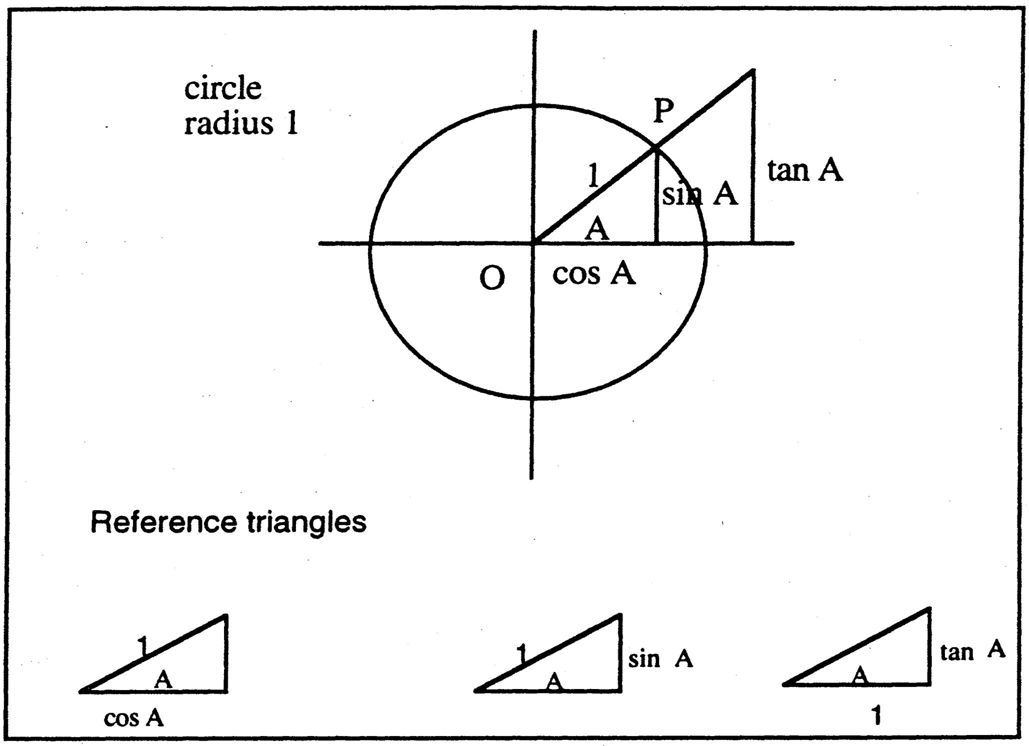


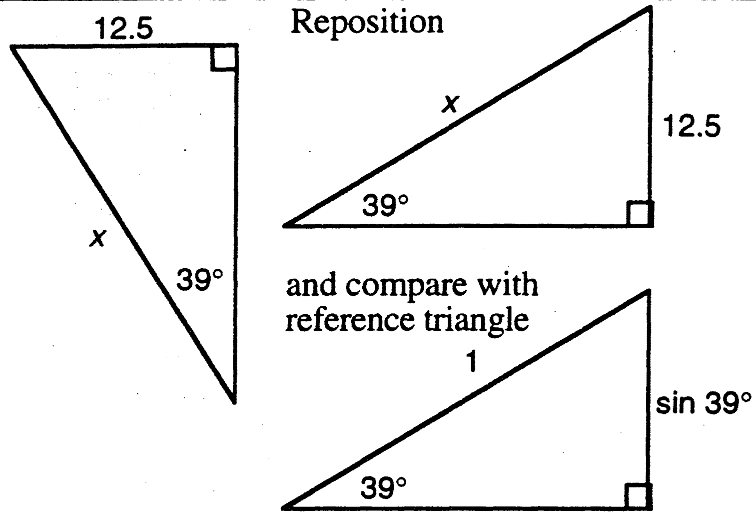
Figure 2. Unit circle definitions of trigonometric functions based on the angle A and reference triangles.

Procedure

An empirical causal-comparative study was undertaken. Classes of Year 10 students at the school where the first author taught were randomly allocated to two groups. Ninety students in four classes were taught basic trigonometry using the ratio method and the other four classes (88 students) were taught by the unit circle method. It happened that the two teachers who taught more than one year 10 class (Teacher 1 and Teacher 2 in Table 1 below) were both allocated one class of each teaching method. This enabled any potential teacher effect to be scrutinised. Each student was assigned to an "ability level" from A (the highest) to F, based on a comprehensive common test taken by all Year 10 students prior to the experiment. A comparison of these rankings showed that the two teaching groups contained a similar number of students at each ability level.

An entire teaching package for each method was prepared by the first author. The teachers were instructed about the methods and the teaching strategies, which included practical work and an outdoor investigation. The emphasis in both cases was on concept development, learning with understanding and enjoyment and skill development. As far as possible the lessons for the two methods were matched in teaching styles and content. The exercises which the students attempted were identical, but ordered differently to fit in with each teaching method. Twenty 45 minute lessons were devoted to teaching this unit of work.

Two tests were administered as both pre-tests and post-tests. Each test contained mathematical items and Likert items to indicate students' attitudes towards the material covered by the test. First, an arithmetic and algebra test was given to determine the level of proficiency of selected algebraic and arithmetic skills which the authors believed to be pre-requisite for successful trigonometry. The items identified students with division misconceptions, students who did not understand the equality nature of the equals sign (Kieran, 1981) and students who could use only informal techniques to solve equations (Kuchemann, 1983).



To find the length of the hypotenuse, x.

Reposition the triangle and match with reference triangle, as shown.

Similar triangles are used to match corresponding sides and set up scale factor

x =scale factor * corresponding side

In this case, x = scale factor* 1
x= 19.863

Figure 3. A difficult example using the unit circle method.

The second test was on trigonometry and was used as pre-test and post-test. It contained 12 questions, each requiring the calculation of a nominated side length in a right angle triangle. About six weeks later, four similar questions were given as part of the end-of-year examination. Other types of questions were treated in the lessons but are not analysed here. Each question on the test was awarded three marks, one for identification of the correct trigonometry function, one for the correct formulation and transposition of the equation to solve (written down or implied by further correct calculation) and one for performing the calculation correctly. If the student initially selected the wrong trigonometry ratio he/she could still gain two marks. Each student therefore received a score out of 36. The students had studied a short unit on trigonometry twelve months previously, using the ratio method. Surprisingly, on the trigonometry pre-test, which was given without warning to the 178 students, all but four students scored zero.

Results

A series of primary comparisons were made to establish which method was more successful. Students' performance on the trigonometry test, and its three constituent skills (their ability to choose the correct function, to formulate the correct equation and to perform the correct calculation) were examined. Two secondary comparisons, improvement in attitude and in solving equations are also reported.

Overall performance on the trigonometry test. Trigonometry performance was defined as the improvement from the pre-test to the post-test. The maximum score was 36. Individual class scores are given in Table 1. The students taught by the ratio method performed significantly better than the unit circle group. This was confirmed by a one-way analysis of variance on the individual scores ($F = 17.79$, $df = 1$, $p < 0.001$). A second measure of success was the higher degree of retention demonstrated by the students on the trigonometry questions on the end-of-year examination. Ratio students obtained a mean score of 8.8 out of a maximum of 12, again shown by an F test to be significantly higher ($p < 0.001$) than the unit circle students' mean score of 7.0. Within each teaching method group, Table 1 shows that the class means show little variation. Factors such as different class compositions and teachers have not influenced trigonometry performance as much as the method of teaching. Further analysis showed that the mean scores of students in every ability group (A to F) was higher for the ratio method than for the unit circle method. Indeed, the lowest mean of the ratio ability groups (27.62 for ability group F) was higher than the highest mean for the unit circle ability groups (25.82 for ability group A).

Choosing the correct trigonometry function. Students taught by the ratio method were more capable of identifying the correct trigonometry functions. The ratio students identified an average of 11.08 of the 12 functions correctly, with a median score of 12. However, the unit circle students identified an average of 8.47 functions correctly, with a median score of 11. A large group of unit circle students identified very few functions. There are several key skills involved. For the ratio method, the student needs to identify the sides in the triangle and then select the correct function. Most students reported using the mnemonic SOHCAHTOA to help them do this. In contrast, the unit circle method requires a complex inter-related set of procedures. The student must re-orient the given triangle if necessary, recall the unit circle diagram and identify which reference triangle matches the re-oriented triangle. Approximately one third of students could not correctly re-orient the triangle and nearly half were unable to draw correct reference triangles. The reference triangle for tan was the most difficult.

Table 1.
Class means (and standard deviations) for trigonometry performance

Ratio Method				Unit Circle Method			
Class	N	Mean	(S.D.)	Class	N	Mean	(S.D.)
Teacher 1	20	31.35	(8.92)	Teacher 1	20	22.60	(9.57)
Teacher 2	23	29.13	(9.59)	Teacher 2	23	19.26	(12.92)
Teacher 3	22	32.32	(6.21)	—			
Teacher 4	21	32.76	(4.98)	—			
—				Teacher 5	22	21.82	(12.64)
—				Teacher 6	25	20.48	(11.05)

Ability to formulate and transpose the equation. The equation was considered to be correctly formulated if it was actually written down or it was implied by the student obtaining the correct answer for the selected ratio. For the ratio method, a correctly transposed equation was expected (i.e. in the form x = . . .) and for the unit circle method the scale factor needed to be determined together with the equation to calculate the side length of the triangle. Questions involving only multiplication were simple to formulate (e.g. find the length of a side opposite a given angle of 20° if the hypotenuse is known). These were completed successfully by 94% of ratio students and 68% of unit circle students. However questions where division was involved (e.g. finding the hypotenuse from the sine of an angle as in Figures 1 and 3) the formulation and transposition was more difficult and the success rate was 77% for ratio students and 38% for the unit circle students. The most common mistake for both methods was to multiply instead of divide. Ratio students were more successful with the division questions than unit circle students were with the multiplication questions. Students from all ability groups were more successful solving the problems involving division using the ratio method. The difference was particularly evident with the lower achieving groups, C to F. These results clearly demonstrate that the ratio method of teaching provides all students and in particular the lower ability students with an effective way to overcome their very real difficulties associated with the solution of more complex equations.

Ability to perform the arithmetical calculations accurately. The difference between the number of correctly formulated equations and the number of correct answers was used as the measure of calculation error. The mean number of errors made was 0.442 (s.d. 1.12) for ratio students and 0.456 (s.d. 0.985) for unit circle students. This difference is not significant ($F = 0.01$, d.f. = 1, $p = 0.931$). Calculation mistakes are not method dependent and are a minor factor in trigonometry errors.

Improvement in attitude. At the same time as the trigonometry pre-test and post-test were given, students were asked to rate their liking for trigonometry on a five point scale. A nine point improvement-in-attitude scale (from -4 to +4) was derived by subtracting the post-test attitude from the pre-test attitude. For the ratio students the mean improvement in attitude was 0.722 (s.d. 1.00), significantly greater than the mean improvement for unit circle students of 0.318 (s.d. 1.15) ($F = 5.75$, df = 1, $p = 0.018$). Every ability level of ratio method students showed a greater improvement in attitude towards trigonometry than the corresponding unit circle ability level students.

Improvement in solving algebraic equations. The scale factor version of the unit circle method is promoted by teachers and texts because it avoids solving equations where the unknown is in the denominator and formal equation solving methods are demanded. On the other hand, the ratio method requires students to solve such equations. Eight items on solving equations with decimals were on the arithmetic and algebra pre- and post- test. Examples are $0.025x = 5$, $30 = 0.003/x$ and $25 = x/0.004$. As expected because of the practice involved, the ratio students showed significantly more improvement than the unit circle students. Ratio students improved by an average of 1.360 questions (s.d. 2.79) and unit circle students improved by 0.402 questions (s.d. 2.26) ($F = 6.17$, df = 1, $p = 0.014$). Again, the greatest improvement was seen with the ratio students in the lower ability groups, C to F.

Conclusion

This study has provided substantial evidence to suggest that the ratio method of teaching introductory trigonometry is a better choice for schools than the unit circle method. Students were better able to master the skills required and, with the greater success, came a greater improvement in attitude to the subject. The advantage the ratio students gained lasted through to the end-of year examination. The mean scores of the four ratio classes clustered together, considerably above the means of the four unit circle classes, showing that the teaching method, not the teacher, was the dominant effect. Students of all ability levels performed better with the ratio method, but the low ability students benefited the most. An analysis of the techniques that made these students in these groups successful is given in Kendal (1992).

Two procedures were critical to success on the test items. The first is an ability to identify which trigonometry function is appropriate for the problem. To identify the correct function many teachers, including those in this study, use a very simple, easily remembered mnemonic SOHCAHTOA. In contrast for the unit circle method there is a set of procedures required for re-positioning the triangle and matching it with a reference triangle. Not surprisingly, since the unit circle method is fraught with multiple opportunities for mistakes, there was a highly significant difference in students' ability to select the correct trigonometry function.

The second procedure involves formulating the correct trigonometric equation and transposing it so that it can be solved. For the ratio method, this is the step where most of the difficulties occur as the students may need to solve complex equations. Proponents of the unit circle method have sought to avoid this by adopting the scale factor method. Sometimes the scale factor is straightforward, able to be readily visualised but sometimes a counter-intuitive division operation is required to calculate it. The results of the experiment show that unit circle students were less successful in identifying the appropriate scale factor in complex cases than ratio students were in transposing the complex equations. Although formulations involving division were difficult for students in both methods, ratio students were more successful. Some transfer of learning, an improvement in the ability of ratio students to solve equations with decimals, was also noted.

Implications for curriculum choices.

There are some clear limitations to this study in assessing the place of the unit circle method in a broader view of school mathematics. In particular, the items on the trigonometry test were skill exercises of only one type, albeit the simplest exercises on the main target of the teaching, solving triangles. Students were not tested for conceptual development, nor to see if foundations had been laid for later extension of the definitions of the trigonometric functions beyond the first quadrant. (Surprisingly, in the unit circle method as advocated by the curriculum guidelines, only first quadrant angles are demonstrated in Year 10.) It is possible that the unit circle students may benefit in the long term in these aspects, but their generally very low facility in what was tested makes this seem unlikely. The shockingly low retention of previous learning, as judged by the pre-test, also indicates caution before choosing any method on the basis of untested long-term benefits.

The unit circle provides concrete meanings for the trigonometric functions, as lengths of intervals which can be directly measured. The problem with the unit circle method is not the unit circle. Rather it is the complex procedure that are required to use the resulting definitions to solve triangles. In recommending the unit circle approach, curriculum developers opted for the more difficult path, with promise but no evidence of any long term benefit. Moreover, the scale factor version of this approach which is designed to eliminate one major difficulty has only replaced it by another of equal or greater magnitude.

Student learning is not a haphazard affair but is controlled by factors such as the method chosen by the teacher to convey the concepts and develop the required skills together with a range of teacher and class influences. We suggest that basic trigonometry be introduced using the unit circle trigonometric function definitions, connecting them to the ratio definitions and then adopting the techniques of the ratio method for the solution of triangles. This approach is adopted by Ganderton, McLeod and Creely in *Mathematics For Australian Schools Year 10* (1991, pp. 258-259) and some other texts. This may enable the younger students to experience real, concrete meanings for trigonometry definitions, lay some foundations for more advanced work and also have the thrill of successful performance in mathematics. It is clear that proposed curriculum change should be backed by empirical research, not only a philosophical debate and innovative teaching ideas.

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