The Riemann Integral in Calculus: Students' Processes and Concepts

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Students' ability to understand calculus has been of concern for some time. Students are often locked into a process-oriented style of thinking which is an obstacle to their understanding of important concepts. In this paper we describe the results of an integral calculus questionnaire given to senior secondary school students and designed to measure their understanding of concepts associated with the Riemann integral. We describe some misconceptions in their understanding and the instrumental, process-oriented thinking underlying them.

Introduction

Calculus reform programmes are now well underway in some parts of the world in an attempt to address the difficulties which many students have had with understanding calculus. Much of the research into these difficulties has uncovered students' conceptual difficulties with limits, differentiation and integration (see e.g. Tall, 1985, 1986a, b; Steen, 1988; Barnes, 1988; Li & Tall, 1993; Thompson, 1994) and considerable effort has also been put into finding ways to improve such understanding, often using computer software such as symbolic manipulators (e.g. Small & Horsack, 1986; Palmiter, 1991; Barnes, 1994; Hubbard, 1995). Heid (1988), for example, suggested giving meaning to basic concepts, by using graphical software and symbolic manipulators to build conceptual insights based on mathematical and cognitive principles. She found that, using these methods, the learning of fundamental concepts was greatly improved in her experimental class. These students made use of a large variety of concept representations and learned how to reason with them. They learned to think for themselves and to construct their own ways of handling the concepts. It became apparent that they had integrated the ideas into their own knowledge structures.

One recurring theme from the research is that students have a tendency to be process-oriented rather than concept-oriented in their approach to calculus. They are much happier following an algorithm or manipulating symbols than they are with dealing with concepts such as limit or rate of change. This is not too surprising when one considers that, until recently, calculus teaching in school and university has often concentrated on process skills rather than conceptual understanding, or, in the words of Skemp (1976), it has been instrumental not relational. A key distinction between relational understanding and instrumental understanding is that the former can be described as learning 'why to', while in contrast, the latter is learning 'how to' and involves learning by rote, memorising facts and rules.

Background

In the research on the calculus there has been a tendency to concentrate on differentiation and its concepts rather than integration. Although this situation is now changing, the purpose of our research is to investigate student thinking and misconceptions when dealing with the Riemann integral. Here, a definition of definite integral can considered whereby a set of n rectangles, which can be freely constructed, is used to approximate the area. This requires that, for a given set of well-defined rectangles, the tops of the rectangles intersect with the graph of the function, the widths of the rectangles partition the horizontal distance involved, and as n‡∞, the width of the rectangles approaches zero. But a definition of the definite integral based on these concepts is often quickly forgotten once the fundamental theorem of the calculus linking areas and antiderivatives is introduced. This is not unnatural, because the Riemann sums used in the definition are tedious to calculate by hand and require the taking, and understanding, of a limit, whereas the antiderivatives involve clear algorithmic processes.

Processes and Concepts

Much has been written in recent years about the relationship between processes and concepts in mathematics. Piaget (1985, p.49) described how "actions or operations become thematised objects of thought or assimilation" and this idea has been further clarified by researchers such as Dubinsky & Lewin (1986) who describe the *encapsulation* of a process as an object. Sfard (1991) too has emphasised that there is a conceptual change when this occurs, involving the conversion of a dynamic process into a static object. Encapsulating both the differentiation and integration processes seems to be an essential prerequisite for understanding the fundamental theorem of calculus. Gray & Tall (1993) have introduced symbolism and define the notion of *procept* as an amalgam of three things - process, symbol and concept. Thus a symbol, such as:

$$\int_1^x \frac{\sin x}{1+\log t}dt$$

evokes both the process of integration and the concept of integral, with the cognitive combination of all three being a procept. Procepts are first met through a process, then a symbolism is introduced for the product of that process, and this symbolism eventually takes on the dual meaning of the process and the object created by the processes. The limit is an important example of a procept in the calculus. For example, both of the symbols: $\lim_{x\to\infty} f(x)$ and $\lim_{n\to\infty} a_n$ may represent either the process of getting close to specific value, or the value of the limit itself. It seems that much of the symbolism used in mathematics carries for the mathematician the dual role of process and concept. This distinction between the usage of symbolisation to stand for a process or a concept or conceptual structure depending on one's point of focus is clearly an important one mathematically. In addition, Gray & Tall (1993) define *proceptual thinking* as the ability to be able to switch one's focus between these dual roles of the symbols as necessary. Someone who has the ability to think in this way may be described as a *versatile* mathematician (Tall & Thomas, 1991). One major difference between the *versatile* mathematician and the learner is that the mathematician already has a global picture of the concept, so that when she/he breaks it down into a number of stages, she/he can see each stage as part of the whole concept. The learner, on the other hand sees only the part in the context of limited understanding.

Method

In the light of the above discussion, our aim is to use computer-based work to improve students' relational understanding of, and proceptual thinking about, Riemann integration. In this aim we agree with Tall (1993), who suggests that the computer relieves the learner of the tyranny of having to encapsulate the process before obtaining a sense of the properties of the object. By using software which carries out the process internally, it may become possible for the learner to explore the properties of the object produced by the process before, at the same time, or after studying the process itself. This new flexibility in curriculum development he has called the *principle of selective construction.*

The first stage of the research, documented here, was to discover what type of thinking and understanding students are developing with current styles of teaching. To this end we developed a questionnaire containing fifteen questions, each of which sought to probe understanding of one aspect of integration, but doing so in a way which often would not enable students to carry out processes or algorithms in order to obtain answers.

This questionnaire on the concepts of the Riemann integral in calculus was given to 47 Form 7 (age 18 years) high school students early in 1996. These students had been studying calculus for about one year and were from 3 schools, one for boys only, one for girls only and one co-educational. Due to advanced promotion through school, the actual ages of the students ranged from 16 to 19 years and there were 16 male and 31 female students.

Questionnaire and Results

The questions we gave to the students produced answers which highlight qualitative differences in thinking. We shall now describe these questions and the responses to them.

Conservation of Integral

The question here was:

Given that $\int_9^{16} \sqrt{x}dx = \frac{74}{3}$, what is $\int_{16}^{9} \sqrt{t}dt$?

This was intended to determine whether students understood the concept of conservation of integral under change of variable and could apply this concept in a situation where the limits were reversed. 34.0% of the students were able to correctly answer this question. We gave the students an actual function, $\sqrt{x}$, to see whether some process-oriented students would calculate the integral(s) in order to answer the question, but in the event, only two students attempted to do this. However, 17.2% of the students invented an algorithm to deal with their lack of understanding of the reversed limits. Since the values of the limits appeared inverted they inverted the answer too and obtained $\frac{3}{74}$. This provides evidence of the way in which a substantial number of students, who are instrumental in their thinking and understanding rather than versatile, conceptual thinkers, attempt to deal with problems which have concepts they do not understand.

The Riemann Integral

A question, dealing with Riemann sums, which also revealed conceptual gaps in understanding was:

If f(x) is an integrable, strictly decreasing function on [1, 5], what is the value of

$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_{i+1})\Delta x - \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x$, where x_i defines the start of the i-th subinterval, and Δx is its width?

(a) very small and positive (b) 0
(c) very small and negative (d) not possible to say
(e) very large and positive (f) very large and negative

This was designed to test understanding of the concept of the equivalence of the limit of the upper and lower sums. Since no function is explicitly stated it was not possible for a process-oriented student to calculate the sums or limits in order to answer the question. A summary of the responses to this question is given in table 1.

Table 1: Summary of question responses for equivalence of the limit of upper and lower sums (N=47)

	No Response	a	b	c	d	e	f
Number of students	19	14	5	5	3	1	0

We see that 40.0% of the students were unable to answer the question (which is admittedly difficult) and only 10.6% answered correctly. The 40.4% of the students who gave a) or c) as the answer do not have the conceptual understanding that the limits of the rightsum and leftsum will be the same, if f(x) is an integrable function. A second question on Riemann sums was:

If f(x) is a strictly decreasing function, an approximation to $\int_1^4 f(x)dx$ using a Riemann sum (i.e. of rectangles under the curve) with left endpoints and number of strips n=10 works out to be 7.615. Will an approximation with left endpoints and number of strips n=50 be:

(a) more than 7.615 (b) less than 7.615
(c) equal to 7.615 (d) less than or equal to 7.615
(e) not possible to say (f) more than or equal to 7.615?

Explain your answer.

Unfortunately only 20 of the students were given this question which looked at the process involved in the area approaching the Riemann limit. Of these 10% gave the correct answer whilst 50% gave no response. Of the others, 30% opted for a) or f) and 10% for d). In spite of being asked to explain, only two students attempted any form of reason for their answer, stating "it was because the curve is lower and wider" and "with left endpoints some might not get included with more strips. So >10 this would have the strips taking up more of the graph." There seems to be a distinct lack of understanding of these types of sums among these students with many students not having encapsulated the process of taking the limit of the sums of the rectangles as an object, namely the definite integral of the function.

Integration and Transformations

We wanted to see if the students had grasped conceptually the relationship between the area under the graph of a function and transformations of the graph. Once again we thought it important to try and move students away from the processes which we thought they would want to use, and so we gave them general integrals in terms of a function f(x) rather than an explicit function which they could integrate directly. The following two questions were given in this area:

If $\int_1^3 f(t)dt = 8.6$, then write down the value of $\int_2^4 f(t-1)dt$ and

If $\int_1^5 f(x)dx = 10$, then write down the value of $\int_1^5 (f(x)+2)dx$.

Students were asked to show their working. Responses to these two questions are summarised in table 2.

Table 2: Summary of responses for transformation questions (N=47)

	Females (N=31)			Males (N=16)		
Transformation Direction	No Response	Correct	Wrong Answer	No Response	Correct	Wrong Answer
Parallel to the t-axis	21	3	7	7	5	4
Parallel to the y-axis	13	2	16	6	4	6

Whilst 17.0% of the students were able to answer the first of these questions correctly, 59.6% were unable to make any response. This could be because they did not recognise the f(t-1) as representing a transformation and there was evidence that this concept was lacking. For example, one student gave 7.6 as the answer, thinking that f(t - 1) implied the value of the integral minus 1, and 5 students (10.6%) attempted to integrate the t - 1 and obtained $t^2 - t$ or $\frac{(t-1)^2}{2}$ leading to a wrong numeric answer, such as 4. Another 2 students (4.3%) who, like those above, were locked into a process-oriented view of the integrals rather than a conceptual one, found another way of introducing a function they could work with. Their idea went as follows:

Let f(t) = ax + b (interestingly, but not surprisingly, a function in x not t)

then $\int_1^3 f(t)dt = 3a + b - a - b = 8.6$ so $2a = 8.6$ and $a = 4.3$.

Thus $\int_2^4 f(t-1)dt = 4a + b - 1 - 2a - b - 1 = 2a - 2 = 8.6 - 2 = 6.6$

The errors in this last line, including miscalculating f(t-1), apart, this demonstrates the lengths students will go to to introduce a known algorithm into a question where they lack the conceptual understanding necessary. In the second of these questions, although 12.8% answered correctly, their preferred method was still to perform an integration, rather than visualise the transformation as creating an extra rectangle of area 4x2 = 8. They usually wrote a version of:

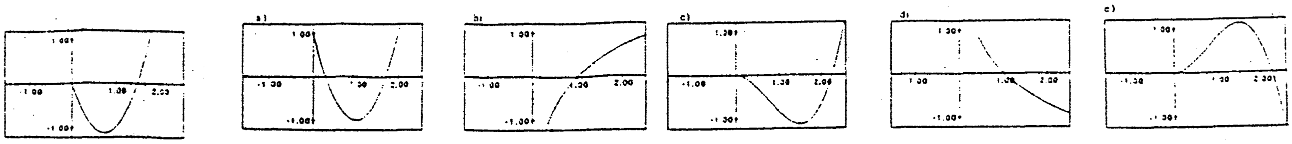
$$\int_1^5 (f(x)+2)dx = \int_1^5 f(x)dx + \int_1^5 2dx = 10 + [2x]_1^5 = 10 + 10 - 2 = 18.$$

There were no significant differences on the basis of gender in the results.

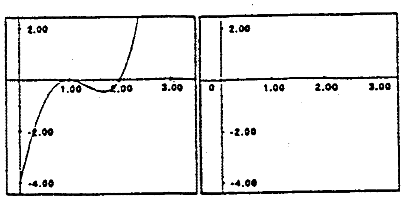
Sketching the Integral Function

We gave the students several questions which tested their understanding of concepts associated with drawing graphs of integral functions from a consideration of area. These were:

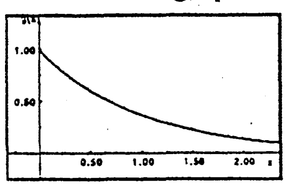
The graph of a function f is shown. Which graph can represent an antiderivative of f? Give your reasons.



The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that F(0) = 0.



Explain why the graph below cannot be the graph of $g(x) = \int_0^x f(t)dt$ for any $f(t)$.



For the first of the questions, 23.4% were successful with several giving data which showed that they had considered a process approach rather than looking at the sign and magnitude of the area under the curve. The results are summarised in table 3. One of the successful students wrote down the function as $x(x - 1.5)$ in order, no doubt, to integrate it. Typical of the thinking used were other successful students who wrote "it must be a positive cubic"; "the antiderivative of f must be a cubic"; and "the antiderivative of f is a ~ shape", again working from a quadratic function with a positive x^2 term and integrating.

Table 3: Summary of question responses for the graph of the antiderivative (N=47)

	No Response	a	b	c	d	e
Number of student responses	10	2	12	11	5	5

In addition 2 students gave other answers

The second question proved rather more testing and, as table 4 shows, only 2 students (4.2%) answered correctly.

Table 4: Summary of responses for the antiderivative sketch question (N=47)

Females (N=31)			Males (N=16)		
No Response	Correct	Wrong Sketch	No Response	Correct	Wrong Sketch
18	2	11	4	0	12

It was again noticeable that a number of the students felt that the only way to answer this question was to find a function which matched the graph, integrate it with respect to x and then sketch that. The concepts of the calculus associating area with integration appear to be little known or applied. Three students, all unsuccessful, wrote down the function $(x - 1)^2(x - 2)$ (one differentiating it) and another wrote $(y + 4)(x - 1)(x - 2)$, confirming that a sizeable number of students rely on the process of integration rather than any conceptual understanding they have built up.

In the third question, 8.5% of the students arrived at the correct solution, but 74.5% were unable to make any attempt at an answer. It may be that they are not used to seeing the concept of the area function written in the form given. We recognise that a written solution gives insufficient insight for this type of investigation and will be interviewing students in the future to ascertain exactly what their difficulties with these types of questions are.

The process-oriented response here was "the function should be with respect to x . . . the graph should be parabola" compared with one student who wrote, very concisely: "$\int_0^0 f(t)\,dt = 0$ not 1", thus demonstrating the power of conceptual thinking.

Conclusions

The evidence we have presented shows that students often lack certain conceptual understanding from current mathematics learning. The high numbers of students who were unable to make any attempt at solving these problems where there is no obvious process to carry out shows that their experiences to date have left important gaps in their conceptual understanding. They have a tendency to see integral calculus as a series of processes with associated algorithms and do not develop the grasp of concepts which would give them the necessary versatility of thought. Thus, instead of having a proceptual view of the symbols in integration they have only a process-oriented view. However, it is not surprising that many students find concepts such as limits and area functions difficult when they are unable to experience these processes directly in many classrooms. It should be possible to design curriculum materials which give an improved cognitive base for a flexible proceptual understanding of limit and other concepts. Thus it may be possible for the student to develop a more balanced view of, say, limits and areas, dually as process and concept by using a computer software. Although the fundamental theorem of calculus doesn't require taking a limit, symbolic manipulation, for example, can allow students to see what is done at each stage, giving them understanding of the concepts as well as the ability to calculate integrals. We hope that the appropriate use of such computer software will improve the students' conceptual understanding of calculus.

One advantage of this method should be that it will engage the student in appropriate activities which require them to think through the process involved. With respect to the graphical aspects of calculus, spreadsheets and symbolic manipulators can be used to develop an understanding of the fundamental concept of calculus, avoiding a passive classroom environment. One of the principal aims of the mathematics educator should be to provide a range of experiences that develop the mathematical ideas in a cognitive manner so that the learner both knows and understands. The computer may focus students on the concepts and ideas of calculus rather than routine computations in a way which may help them to gain understanding which our survey has shown to be lacking.

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