Fairness and Fractions in Early Childhood

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Responses of 24 children from pre-Grade 1 to Grade 4 were used to illustrate the range of understanding of the partition of continuous fractions in the early years of schooling. The SOLO developmental model with multimodal functioning was used to classify responses and it is hypothesised in this paper that ikonic and early concrete symbolic functioning are exhibited in these grades as fairness becomes a mathematical criterion for work with fractional parts. Implications are drawn for further research and for the classroom.

While there has been a move to diminish the importance of operations with fractions in the mathematics curriculum (Australian Education Council [AEC], 1994), there is still an acknowledged need to build an understanding of what the common fractions are and how they can act as operators on sets or wholes (e.g., AEC, 1994, pp. 40, 56). Often associated with the division of a whole into parts is the idea of fairness. Adults readily accept the nature of a partition being fair if the parts are of an equal size. Discussions of fairness in relation to fractions with young children, however, may not be based on the same shared understanding of the term fair.

While research related to young children's understanding of fractions continues to be detailed (e.g., Saenz-Ludlow, 1994, 1995) and the links are made for teachers with literature at this level (e.g., Conaway & Midkiff, 1994), very little explicit acknowledgment is made of the contribution that the intuitive idea of fairness makes to initial efforts to construct partition of wholes. When mentioned (e.g., Peck & Connell, 1991), it is often as an afterthought since the assumption that children understand what teachers mean seems nearly universal. A notable exception is the work of Streefland (1991) who acknowledges the existence of alternative frameworks by telling a story of a mother who "divided one or two apples among eight children and only the grown-ups had the right to a whole apple" (p. 63). He then develops in his class of fraction researchers a climate where they construct a common shared understanding of fairness in order to solve fraction problems.

The data reported in the present study were collected as part of a larger study of fraction and decimal understanding of students from pre-Grade 1 to Grade 10 (Watson, Collis & Campbell, 1995). The problem chosen for analysis here was used in interview settings with concrete materials. The objective was to document student understanding of the sharing process involving three equal parts in a context which involved continuous partitioning of wholes.

Theoretical model

The model used to analyse the data was the SOLO model with multimodal functioning (Biggs & Collis, 1982; 1989; Collis & Biggs, 1991). This model has been successfully applied to the analysis of student responses to tasks associated with volume measurement (Campbell, Watson & Collis, 1992) early multiplication word problems (Watson & Mulligan, 1990), and the search for associations in a data set (Watson, Collis, Callingham & Moritz, in press), as well as fractions and decimals (Watson, et al., 1995). The model, which grew out of the Piagetian tradition, postulates five modes of functioning which originate in a fixed order but which continue to develop alongside each other throughout life: sensorimotor (from birth), ikonic (from early childhood), concrete symbolic (from the years of schooling), formal (for those capable of higher education), and post formal (associated with research). Of particular interest in this study are the ikonic and concrete symbolic modes, these being associated respectively with intuitive functioning and with the symbolic learning which takes place in school based on concrete materials, and their influence on thinking about fractions. As well, within each mode, there are developmental sequences which are hierarchical and are observed in cycles of response. Each cycle contains:

(i) Unistructural responses (U) where individual skills or constructs can be used separately only;
(ii) Multistructural responses (M) in which several such skills or constructs can be used, usually in sequence; and
(iii) Relational responses (R) in which there is a coordinated mastery of the set of component skills or constructs.

The research of Watson, et al. (1995) observed two UMR cycles in the concrete symbolic mode in relation to fractions, the first building the basic concept and the second applying it in problem solving situations. The work of Watson and Mulligan (1990) observed a UMR cycle in the ikonic mode when young children were first introduced to a basic word problem involving the concept of multiplication, leading to one in concrete symbolic mode for those who were taking on the concrete symbolic idea of repeated addition. A further study by Watson, Campbell and Collis (1993) found evidence for functioning in both ikonic and concrete symbolic modes, in this case with multimodal functioning occurring with "bottom up" support from the ikonic mode to assist problem solving in the concrete symbolic mode. The current study adds to the picture in the development of understanding of fractions by hypothesising an ikonic UMR cycle as well as ikonic support for functioning in the early part of the concrete symbolic mode.

Method

Three children from each of pre-Grade 1 and Grade 1 and six children from each of Grades 2 to Grade 4 were interviewed for approximately 45 minutes each. Concrete materials were provided for the children to use in solving the problem and were chosen to access the children's out-of school experiences. For the Pancake Problem described here, children were given a pancake and asked to cut it so that it could be shared fairly among three dolls. Actual dolls and pancakes were used rather than school-based materials such as blocks and diagrams. At each pause the child was asked if the sharing were fair. The interviews were tape recorded and notes taken by a trained research assistant. Responses were then summarised and similar responses grouped together. The three authors then independently classified the responses in the light of theoretical SOLO model with multimodal functioning. The order of rankings were very similar and discussion in relation to the model resolved the minor discrepancies. The names of all students were changed for this report.

Results

The individual responses will be discussed separately and then an overall model postulated for the development of fractional understanding by young children. In the Pancake Problem there are two issues associated with fair fractional sharing. One is the number of pieces given to each doll and the other is the size of the pieces. The approaches to solving these dilemmas present the bench marks of development of understanding. The figures are stylised computer simulations of the students' actions.

The Ikonic Mode

To be considered as responses in the ikonic mode responses had to acknowledge an appreciation of the splitting task associated with sharing. The conservation of number (typical of the concrete symbolic mode) necessary to distribute parts to the dolls as required by the task, however, was not present.

At the unistructural level of the ikonic mode (IK-U) the response of Josh indicated a single splitting action, cutting the pancake in two parts. These were given to two of the three dolls. The concept present is that of sharing but it cannot be carried out to satisfy the constraints of the problem.

At the multistructural level (IK-M) Jan's response indicated a more complex attempt. She said "three" but in making three cuts of the pancake produced four pieces , did not know what to do with them and hesitated in confusion. In this case a more complex splitting was attempted but the dilemma of sharing could not be resolved.

At the relational level (IK-R) Yvonne also cut the pancake into four roughly equal pieces. She resolved the sharing by giving two pieces to one doll and one to each of the others. At this level, the necessity to split the pancake and share it among all the dolls was understood to the extent that all dolls did receive some pancake. The conservation of number of pieces or quantity required for fair sharing, however was not present.

All of these students were subsequently asked if what they had done was fair. All thought for a while, replied "no" and made another attempt described below. The prompting to consider fairness and the presence of an older "teacher" model, may illustrate the relevance of Vygotsky's suggestion of a zone of proximal development and Lamborn and Fisher's (1988) research supporting the benefit of prompting to achieve optimal rather than functional responses.

The Concrete Symbolic Mode

To be classified in the concrete symbolic mode responses had to exhibit a sharing process which at least demonstrated an appreciation of the need for conservation in terms of the number of pieces given to each doll. As the Pancake Problem was a continuous rather than discrete sharing problem, number of pieces alone was not a sufficient criterion for fair sharing without a consideration of the size of the pieces. All subsequent responses except one were considered to be in the first cycle of the concrete mode indicating the building of the fraction concept (cf. Watson, et al., 1995).

Unistructural level: Six students produced responses considered at the unistructural level (CS-U_1). These exhibited a sharing based on conservation of number, not size, and if there were leftover bits from the distribution this did not create a conflict for the student. Alan's distribution shown in Figure 1 began with "quarters", of which he distributed one to each doll. He did not know what to do with the leftovers but having carried out a single sharing operation, was happy he had fairly distributed the pancake.

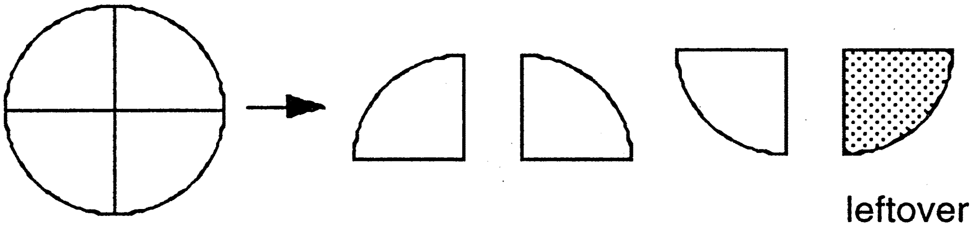

Figure 1. Alan's sharing.

Barb, Karen, Jan (second attempt) and Yvonne (second attempt) each cut the pancake differently but in such a way that each doll received two pieces of unequal size. Karen and Jan each had leftover bits as well, with Jan hiding her spare bit under the table. Jan's action was likely ikonic in nature so that the dolls could not see the part they did not receive. There was no indication that the piece was being hidden from the researcher or that Jan realised a conflict in relation to the distribution of the whole. Each claimed that the sharing was fair because the dolls had two pieces of pancake each. Barb's sharing is shown in Figure 2.

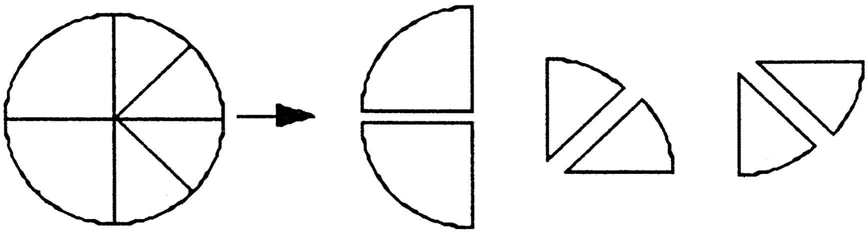

Figure 2. Barb's sharing.

Carla cut three similar pieces from the pancake, then cut the remainder into three very unequal sizes. In distributing the pieces she gave two to each with the largest piece to the doll in the middle, saying "she can decide if she wants to share it." This indicates ikonic support, in including reference to the dolls in the response. Although it may have been the beginning of an acknowledgment of the need for equal sized portions, it was not resolved in the concrete symbolic mode but by reverting to an intuitive ikonic solution.

Multistructural level: The benchmark of responses at the multistructural level (CS-M_1) was the realisation that having the same number of pieces is not enough to constitute fair sharing and that left overs are not possible when wholes are shared fairly. At this level, however, ad hoc methods were used, usually in sequence, to determine fair shares. Kara for example cut the pancake roughly in half and then cut the larger half again. When these pieces were not to her satisfaction she continued making adjustments roughly as shown in Figure 3. Josh's second attempt following his cut into halves was similar. In these cases dolls did not always have the same number of pieces.

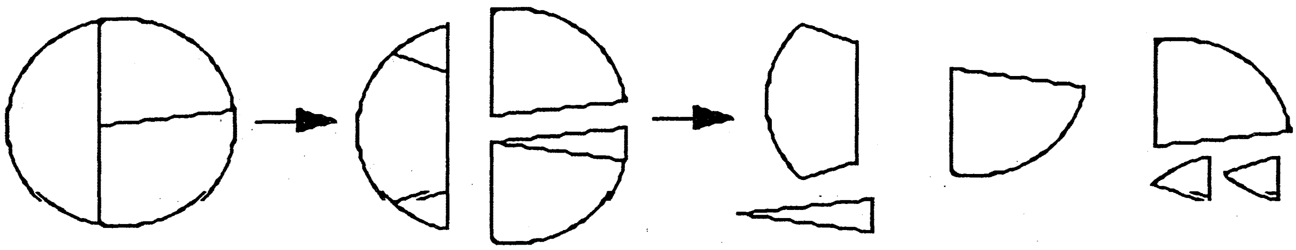

Figure 3. Kara's sharing.

Ned began with cuts similar to "eighths", giving two to each doll. When he realised he had two pieces leftover he retrieved some and adjusted size and sharing until he was satisfied each had the same amount. Eight students began with two roughly equally spaced parallel cuts across the pancake. When questioned on the fairness of the three shares, seven said yes, they were fair because they were the same width. Dan, however, whose cutting was not quite as accurate as the others, cut bits of the corners off his outside pieces to make the shares more fair. His method is shown in Figure 4. At this level the non-integrated nature of adjustments is observed.

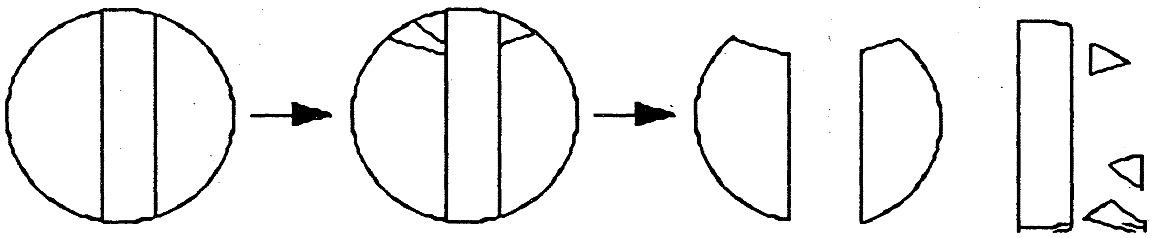

Figure 4. Dan's sharing.

Relational level: At the relational level (CS-R_1), responses displayed an integrated strategy, showing an overview of the problem and an insight into the measurement principles involved and hence beginning to incorporate geometric principles in the consideration of what a fair share should be. Some of these were relatively complex because the task was related to thirds. All solutions however, resolved the necessity to provide both an equal number of pieces and pieces of the same size, totalling the same amount to each person. Carol, Ron and Mark first divided the pancake into quarters (half twice) and then divided one quarter into equal thirds. Each doll was then given one large and one small piece, as shown in Figure 5.

George started in the same manner as the previous three, but then cut the final quarter into six equal pieces, resulting in each doll receiving one large and two small pieces. Saul again began with quarters, splitting three of these into halves and the fourth into thirds. This produced nine pieces to be distributed, with two relatively larger and one smaller, to each doll. These solutions indicate the repetitive use of geometric division into parts (although the terminology was not used) to solve the problem, typical of those who can relate the sharing and the need for equal parts using a relatively complex method. Tara proceeded in two stages but avoided quarters by using thirds at the second stage. Her solution is shown in Figure 6.

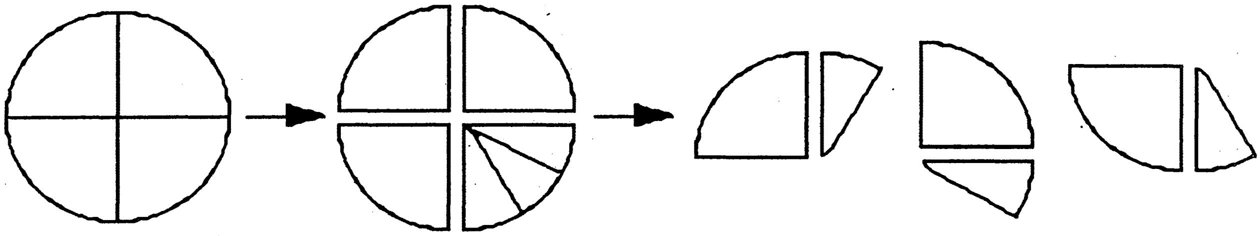

Figure 5. Carol's, Ron's and Mark's sharings.

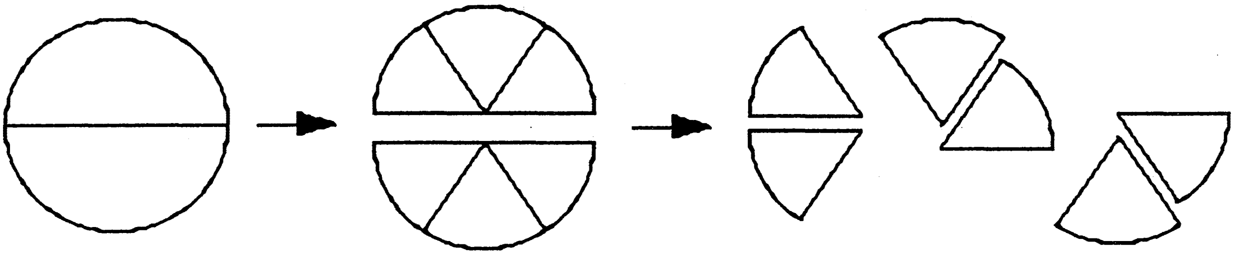

Figure 6. Tara's sharing.

Finally at the relational level, Kay and May (second attempt) realised that parallel cuts of equal width would not produce equal portions and discussed the necessity to make the middle section thinner to compensate. Their solutions are shown in Figure 7. Although it was not possible for these students to measure accurately, the responses reflected a higher level of understanding and relating of geometric principles to the problem than did the other responses involving parallel cuts.

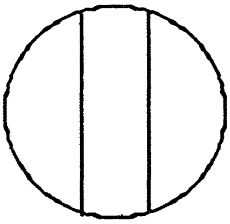

Figure 7. Kay's and May's sharings to compensate for circular shape.

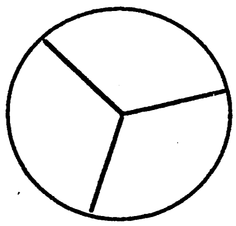

Figure 8. Josh's final sharing.

Unistructural level, second cycle: There was one response which was judged to have reached the unistructural level of the second cycle of the concrete symbolic mode (CS-U_2). Although only allowed one pancake to solve this problem, Josh was subsequently given another pancake for a different problem. This he immediately cut into thirds as shown in Figure 8 to show what he wanted to do now for the earlier sharing problem. This solution is typical of all students above Grade 4 who were interviewed as part of the larger study and indicates again the possibility for optimal performance and learning during a one-to-one interview. It appears that the more complex relationships needed to solve the problem in the first cycle of the concrete symbolic mode are replaced by a single, new, more sophisticated concept.

Summary of results

Table 1 summarises the responses of the 24 children in this study by SOLO level and by grade level. The increased sophistication by grade level is evident, but it is also evident there is potential for a great variety in responses, particularly in Grades 1 and 2.

Table 1.
Summary of responses by SOLO level and by grade level

	Pre-Grade 1	Grade 1	Grade 2	Grade 3	Grade 4
IK-U single idea - sharing			Josh - half→ 2		
IK-M divides into parts inappropriately	Jan - 4 pieces, 3 dolls confusion				
IK-R sharing to all but not conservation		Yvonne - 4 bits, shares 2, 1, 1			
CS-U$_1$ sharing based on conservation of number not size - left over bits not a conflict	Barb - 2 each, unequal Jan - after repeated cuts, 2 each hide 1 unequal [+IK] Alan - 1 each 1 leftover	Yvonne - [after 'fair' prompt] 2 each, unequal	Carla - 3+3 pieces, 2 each, bigger for middle doll [+IK] Karen - 2 unequal pieces template for 3 but leftovers		
CS-M$_1$ ad hoc adjustment for equal size -uses appearance		Ned - near $\frac{1}{8}$ths then adjusts	Josh - half of half, smaller and smaller Cam - judges equal width (parallel) Dan - judges parallel then compensates	Kara - half, half then adjust; Alf, Marg - judge equal width (parallel)	Carl, Judy, Owen, May - judge equal width (parallel)
CS-R$_1$ geometric 'principles' to adjust for equal size		Carol - 2 each; 1 *l*, 1*s*	Ron - 2 each; 1 *l*, 1*s*	George - 3 each; 1 *l*, 2 *s* Tara - 2 steps, 2 equal each Saul - 3 each; 2 *l*, 1 *s*	Mark - 2 each, 1 *l*, 1 *s*; Kay, May - parallel cuts; middle thinner
CS-U$_2$ consolidation			Josh - $\frac{1}{3}$s "as a pie"		

The salient features which appear to distinguish the development of fair sharing in relation to continuous fractions in early childhood support the following SOLO development in the ikonic and early concrete symbolic modes.

IK-U There is a single idea of sharing or partitioning into two parts only.

IK-M The idea of sharing can cater for more parts but resolution is difficult in cases such as four shares for three dolls. Conceiving the task but not being able to resolve such conflict successfully is typical of multistructural responses.

IK-R Sharing is understood in more complex environments (such as three dolls) but without conservation the distribution which takes place is inconsistent with the mathematical condition set in the problem. Hence at this level while a resolution is reached using an ikonic (and mathematically idiosyncratic) notion of fair sharing the lack of conservation precludes the response from the concrete symbolic mode.

CS-U_1 The criterion for fair sharing at this level is based on conservation of number and requires that a fair distribution is based on the same number of pieces (for each doll). Any conflict relating to the size of shares is not acknowledged and leftover parts of the whole are likely to be ignored.

CS-M_1 Sharing at this level recognises the need not only for an equal number of parts but also for an equivalent amount (of pancake) in each share. The method of achieving this outcome is ad hoc, often requiring a sequence of attempts and estimation. Appearance may be an important factor.

CS-R_1 The realisation that sharing requires equal portions is also the significant feature at this level but the method of achieving it utilises other mathematical principles (e.g., geometry and measurement) which are integrated and applied in a coherent fashion. Elementary geometric divisions of the circle starting with halves and/or compensating for the width of parallel cuts are typical examples.

CS-U_2 The consolidation of the idea of third in relation to a circle and equal sharing affords the opportunity to cut the circle directly from the centre in "pie-shaped" pieces. The need for a more complex relationship, typical of the previous level, is replaced by a single simplified concept.

Discussion

The importance of these findings relates both to our understanding of how mathematical ideas develop in young children and to suggestions for classroom teachers. These will be considered in turn.

The response of students at the CS-M_1, level support results obtained by Watson, et al. (1993) in relation to the Mars Bar Problem. In that problem students were asked if a Mars bar cut in half lengthways or widthways would produce equal shares. There the multistructural level was reached when students could visually or physically reorganise one shape of half a bar to fit another shape of half a bar and accept the fairness of the shares. The context of that problem meant that students who did not consider that the shape of a half could vary were classified as CS-U_1 . This is consistent with the lack of consideration of size in the context of this study. The "bottom up" ikonic support observed in the previous study also occurred here.

The progression of attempts to share in the ikonic mode which fall short of mathematical fairness are similar to those of Watson and Mulligan (1990) in relation to an early multiplication problem. Although there is some understanding of the task and the ability to manipulate concrete materials in the general context, lack of conservation means that the task cannot be completed in the way required by the mathematics curriculum.

The relationship of fairness to other topics in the mathematics curriculum, such as chance, would indicate that early concepts should be compared in different settings. The work of Lidster, Pereira-Mendoza, Watson and Collis (1995) suggests an ikonic cycle of fairness in relationship to dice which grows from being ego-centric to other-oriented to world-view based. This is followed by a concrete symbolic construction which grows out of considering physical features of dice. Combining the analyses from these studies and further research may help us develop a better understanding of the ideas on fairness which children bring to school and how they may be fostered to become a useful mathematical construct.

It is clear that when teachers discuss fairness in classroom settings they need to be aware of the multiple understandings which may be present. Detailed questioning of responses which accept a fractional sharing as fair may help find the base lines for tasks to aid conceptual development. At least two important aspects arise from this research. One relates to leftovers which may remain from a whole when fractional parts are divided. While in an out-of-school setting it may be entirely reasonable to have leftover pieces of cake, in later mathematics the division of a whole into equal parts will implicitly assume that there are no leftovers. This distinction will need to be made for some students as they begin work with fractions.

The other aspect relates to the discrete and continuous nature of partition possible of a whole. For discrete sets of equal-sized objects, number of pieces is adequate for determining fair shares. For continuous wholes, however, the amount or portion of the whole is the salient feature to be considered. Fair sharing may not depend on the number of pieces in the share, but the amount of the whole must be the same. For the students in this study, a combination of approaches was the usual response. Ideas such as those of Streefland (1991) which build a consideration of fairness into initial work on fractions may be very helpful in confronting difficulties and developing consistent definitions in the classroom.

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References

Australian Education Council. (1994). *Mathematics: A curriculum profile for Australian schools*. Carlton, Vic.: Curriculum Corporation.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning: The SOLO taxonomy*. New York: Academic Press.

Biggs, J. B., & Collis, K. F. (1991). Multimodal learning and the quality of intelligent behaviour. In H. A. H. Rowe (Ed.), *Intelligence: Reconceptualisation and measurement* (pp. 57-76). Hillsdale, N. J.: Lawrence Erlbaum.

Campbell, K. J., Watson, J. M., & Collis, K. F. (1992). Volume measurement and intellectual development. *Journal of Structural Learning, 11*, 279-298.

Collis, K. F., & Biggs, J. B. (1991). Developmental determinants of qualitative aspects of school learning. In G. Evans (Ed.) *Learning and teaching cognitive skills* (pp. 185-207). Melbourne: Australian Council for Educational Research.

Conway, B., & Midkiff, R. B. (1994). Connecting literature, language and fractions. *Arithmetic Teacher, 41* (8), 430-434.

Lamborn, S. D., & Fischer, K. W. (1988). Optimal and functional levels in cognitive development: The individual's developmental range. *Newsletter of the International Society for the Study of Behavioural Development,* No. 2 (Serial No. 14), 1-4.

Lidster, S. T., Pereira-Mendoza, L., Watson, J. M, & Collis, K. F. (1995, November). *What's fair for Grade 6?* Paper presented at the Annual Conference of the Australian Association for Research in Education, Hobart.

Peck, D. M., & Connell, M. L. (1991). Using physical materials to develop mathematical intuition in fraction part-whole situations. *Focus on Learning Problems in Mathematics, 13*(4), 3-12.

Saenz-Ludlow, A. (1994). Michael's fractions schemes. *Journal for Research in Mathematics Education, 25*, 50-85.

Saenz-Ludlow, A. (1995). Ann's fractions schemes. *Educational Studies in Mathematics*, *28*, 101-132.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm for developmental research*. Dordrecht: Kluwer Academic.

Watson, J. M., Campbell, K. J., & Collis, K. F. (1993). Multimodal functioning in understanding fractions. *Journal of Mathematical Behavior, 12*, 45-62.

Watson, J. M., Collis, K. F., & Campbell, K. J. (1995). Developmental structure in the understanding of common and decimal fractions. *Focus on Learning Problems in Mathematics*, *17*(1), 1-24.

Watson, J. M., Collis, K. F., Callingham, R. A., & Moritz, J. B. (in press). A model for assessing higher order thinking in statistics. *Educational Research and Evaluation.*

Watson, J. M., & Mulligan, J. (1990). Mapping solutions to an early multiplication word problem. *Mathematics Education Research Journal, 2*(2), 28-44.