Constructing Initial Algebraic Understanding: The Proposed Research.

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An initial framework to be used as part of a study that is exploring how students construct initial algebraic understanding is proposed. As a starting point, Pirie and Kieren's dynamical model of growth of mathematical understanding is proposed. A qualitative ethnographic case study approach to be used in the research is described.

Introduction

The Concepts in Secondary Mathematics and Science [CSMS] project (1975-79) investigated students' understanding of letters (Kuchemann, 1981). Approximately 3500 students were interviewed and tested by means of a "pen and paper" test. As a result, Kuchemann reached the conclusion that many students' progress in algebra was limited by the fact that they never moved beyond viewing letters as numerical placeholders. A follow-up project, Strategies and Errors in Secondary Mathematics [SESM] (1980-83), reinforced these findings (Booth, 1984) when it investigated the reasons underlying errors made in school students' algebra.

These projects and subsequent studies highlight that research into teaching and learning of beginning school algebra is of major interest. Some of this research has focussed on the transition between arithmetic and algebra. Concepts such as cognitive gap (Herscovics & Linchevski, 1994) and didactic cuts (Filloy & Rojano, 1989) have been proposed to explain some of the difficulties and obstacles that students encounter in developing algebraic concepts. One particular component of such research is how the initial concepts of school algebra may be taught more meaningfully. A number of alternatives have been proposed. These differ from "chalk and talk" methods that seem merely to manipulate letters. Some of these newer proposed approaches focus on expressing generality in the context of everyday situations through the use of patterning activities and emphasise that algebra is a language (Mason, Graham, Pimm, & Gowar, 1985). Other approaches introduce algebraic concepts by using manipulatives such as match sticks, counters and cups in conjunction with pattern building (Lowe, Johnston, Kissane, & Willis, 1993; MacGregor, Pegg, Redden, & Stacey, 1994; Quinlan, Low, Sawyer, & White, 1987). Still other programs of introducing algebra indicate that functions should be present in algebra teaching from the outset and not left till late in the instructional sequence (Yerushalmy & Schwartz, 1993).

The Research

The literature highlights the need for more inquiry into students' thinking in the area of beginning algebra. How students construct their initial algebraic understandings is one question that continues to seek explanation. The research proposal aims at contributing to the cognitive research literature in this area of early algebraic concept formation. It anticipates exploring how students construct their initial algebraic understanding and aims at developing a theoretical framework to explain such construction. It attempts to investigate the mathematical thinking of individual students in the area of beginning algebra. In order to achieve this it is necessary that the design of the research be different from that of those which applied batteries of pen-and-paper tests to large numbers of students. The research design focuses on monitoring a small group of students for an extended period of time. This research, therefore, tracks a group of about ten students over a two year period as they proceed from the primary school (year seven) to the secondary school (year eight). It is in year eight that formal school algebra is introduced to them for the first time. The setting for this research is the naturalistic setting of the classroom.

A Case Study Approach

Since this study attempts to provide a fine-grained analysis of how students construct their initial algebraic understanding, an ethnographic case study appears appropriate. In this research, case study is chosen since it is the approach that is called for "when the phenomenon under study is not readily distinguishable from its context" (Yin, 1993). The research effort of this study is directed at describing and explaining students' understandings in the context of the classroom itself and will be qualitative in nature. How students construct their initial algebraic understandings is of major concern to this research and it is for this reason that case study is chosen as the preferred strategy. Additionally, the researcher is entering the naturalistic setting of the classroom and consequently has limited control over the events. "Case studies are the preferred strategy when 'how' or 'why' questions are being posed, when the investigator has little control over events, and when the focus is on contemporary phenomenon within some real-life context" (Yin, 1994, p. 1).

Examining, categorizing, tabulating, or otherwise recombining the evidence to address the initial propositions of a study is the essence of data analysis. Analysis of case study evidence, unlike statistical analysis, has few fixed formulas to guide the researcher. "Much depends on the investigator's own style of rigorous thinking, along with the sufficient presentation of evidence and careful consideration of alternative interpretations" (Yin, 1994, pp. 102-103). Since "the ultimate goal is to treat the evidence fairly, to produce compelling analytic conclusions, and to rule out alternative interpretations" (p. 103), it is important to have a general strategy. Two such types of general strategy, as identified by Yin, are relying on theoretical propositions and developing a case description.

The first of these two general strategies, relying on theoretical propositions, applies when theoretical propositions lead to the case study. The objectives and design of the case study are based on these propositions, which in turn invite research questions. These propositions may even lead to novel insights. They guide the data collection and the case study analysis. Data appropriate to the building and growth of algebraic understanding of students becomes the focus in this research on algebraic concept formation. "Theoretical propositions about causal relations - answers to 'how' and 'why' questions - can be very useful in guiding case study analysis" (Yin, 1994, p. 104). The second general strategy, developing a case description, involves developing a descriptive framework for organising the case. It generally serves as an alternative when theoretical propositions are absent.

In this research, the first of these two general strategies, relying on theoretical propositions, is the most appropriate one. The personal experience of the researcher in the classroom, teaching introductory algebra to year eight students contributed to an original theoretical proposition that the use of concrete representations may be a useful instructional method of introducing algebra to students in a meaningful way. This in turn has helped formulate the aims of this research. They are to explore how students construct their initial algebraic understanding as well as develop a theoretical framework to explain this construction. The results of this research hopefully will contribute to informing future teaching practice in the area.

Within this general analytic strategy, specific analytic techniques need to be considered. Two sets of such specific analytic techniques have been identified (Yin, 1994). They are dominant modes and lesser modes. The dominant modes are: pattern-matching, explanation-building, time-series analysis, and program logic models. The lesser modes of analysis are: analysing embedded units, making repeated observations, and doing a case survey (secondary analysis across cases). These lesser modes need to be used in conjunction with one of the dominant modes.

The pattern-matching mode of case study analysis compares an empirically based pattern with a predicted one. Explanation-building, the second analytic strategy, is a special type of pattern-matching. "The goal is to analyze the case study data by building an explanation about the case" (Yin, 1994, p. 110). This is mainly relevant to explanatory case studies. The third analytic strategy is directly analogous to the time series analysis conducted in experiments and quasi-experiments. This strategy, time-series analysis, traces changes over time and may be possible "if the events over time have been traced in detail and with precision" (Yin, 1994, p. 113). The last analytic strategy is that of program logic models and is a combination of pattern-matching and time-series analysis. "The analysis deliberately stipulates a complex chain of events (pattern) over time (time series)" (Yin, 1994, p. 118).

It is expected that within the current research the main dominant analysis mode will be explanation-building. Being mainly relevant to explanatory case studies, this mode of analysis is very appropriate for this research. Glaser and Strauss (1967) have suggested a procedure for exploratory case studies that is similar to this analysis mode. However, the goal of explanation-building is to conclude the case while for Glaser and Strauss it is to build and develop ideas for further study. The final explanation may not be fully stipulated at the beginning of this explanation-building mode of analysis and in this regard differs from pattern-matching. "The case study evidence is examined, theoretical positions are revised, and the evidence is examined once again from a new perspective, in this iterative mode" (Yin, 1994, p. 111). As the data are collected and examined in this study, it may be appropriate that, in addition to explanation-building, other modes of analysis may be used according to the nature of this data collected. Of the lesser modes of analysis within the general strategy, it is expected that making repeated observations will be a useful technique. The data may also determine the need to analyse embedded units.

There are four principles, independent of the specific analytic strategies chosen, that ensure the highest quality of the analysis in any case study. The analysis should show

- that it relied on all the relevant evidence,
- that it included all major rival interpretations,
- that it addressed the most significant aspect of the case study, and
- that the researcher brought his or her own prior, expert knowledge to the case study (Yin, 1994).

The analysis of this research will seek to ensure that all of these principles are implemented. Data will be collected by means of student and teacher interviews, student journals, teacher diaries, and participant observation.

An Initial Framework

As an initial framework for investigating the growth of understanding in the concepts of variable and function, the model that Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) have posited is an appropriate one because of its dynamical nature. This framework is necessary in order to guide observation and analysis as the research proceeds. In summary the essential features of this model are indicated in figure 1 below:

A Dynamical Model of the Growth of Mathematical Understanding

In this dynamical model of the growth of mathematical understanding, Pirie and Kieren (1992) have proposed eight potential levels of understanding for a *particular person with respect to a particular topic*. "It is, in essence, a theory and a model of personal growth" (p. 245). They believe that there is "no such thing as understanding in the abstract" (Pirie & Kieren, 1994a, p. 39). Mathematical understanding is seen as "a process, grounded within a person, within a topic, within a particular environment" (p. 39). A brief overview of these eight levels is given.

Pirie and Kieren (1992) use the term *primitive knowing* to describe the starting level in this process of coming to understand any mathematical topic. It simply indicates the starting point and does not imply low level mathematics. It is concerned with the assumptions that the teacher may make with regard to the prior knowledge that a particular student has before beginning a new task. The observer is not able to know what this primitive knowing really is.

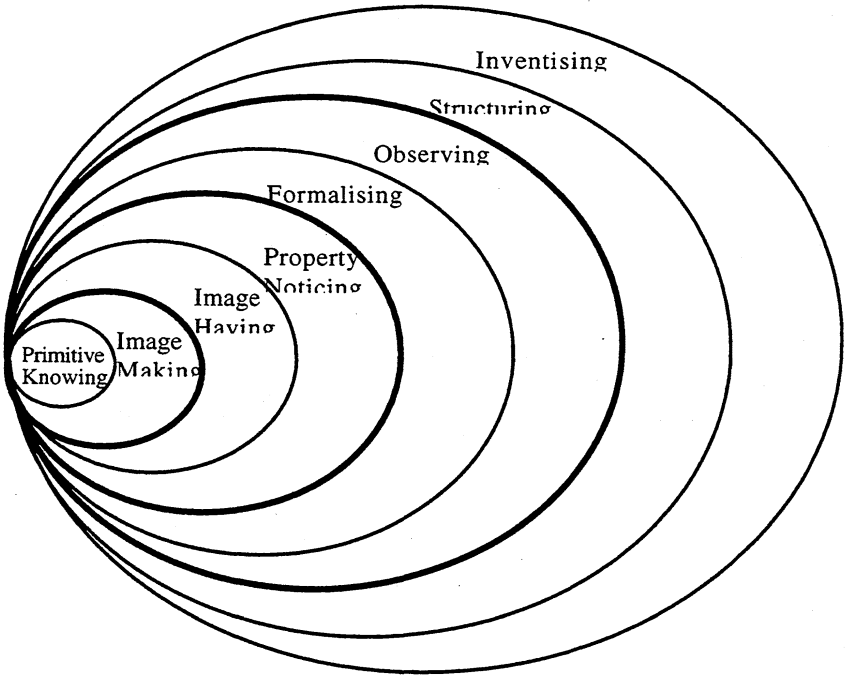

Figure 1: Model of a dynamical theory of the growth of mathematical understanding. (Kieren & Pirie, 1992, 1994; Pirie & Kieren, 1989, 1991, 1992, 1994a, 1994b)

The second level, called *image making*, is the level at which "the learner is asked to make distinctions in previous knowing and use it in new ways" (Pirie & Kieren, 1994b). It is through specific tasks that the student makes distinctions in previous abilities using them either to new ends or under new conditions. The term "image" here is used in a broad sense. It is not used merely for pictorial representation but for any kind of mental representation.

These singular, directed activities of the image making level are replaced by a mental object at the next level, *image having*. At this level, a student's mathematics is freed from the need to perform particular actions. "The image itself, as a mental object, can be used in mathematical knowing" (Pirie & Kieren, 1992).

A fourth mode of understanding occurs "when one can manipulate or combine aspects of ones images to construct specific, relevant properties" (Pirie & Kieren, 1994b). This involves making connections and noting distinctions between images. Combinations of images can occur. Because ones images are examined for specific properties in order to make these connections, distinctions and combinations, this fourth level is referred to as *property noticing*. It is at this level that relationships are recorded.

Formalising is the next level of understanding activity. "The person abstracts a method or common quality from the previous image dependent know how which characterised her [or his] noticed properties" (Pirie & Kieren, 1994b). This level of understanding entails thinking about the noted properties and informal patterns in the previous level and abstracting commonalities. Now the person has a mental object that is class-like and is not dependent on meaningful images.

Having established these formalisations, the students are in a position to reflect on them and reference their own formal thinking. Thus, the next level is called *observing* and involves noticing one's formalising and organising these observations. A deeper understanding is then gained at the next level where these formal observations are able to be explained in terms of a logical structure. Thus, this seventh level or mode of understanding is known as *structuring*. It means that the student is aware of his or her assumptions and the sequence of the observations made at level six. Interrelationships between observations are established logically. "Structuring occurs when one attempts to think about ones [*sic*] formal observations as a theory" (Pirie & Kieren, 1994b, p. 171). The student is aware how the set of theorems is inter-related. Justification and verification of statements occurs at this level. Formal mathematical proofs are used in this mode of mathematical understanding. It may be thought of "as setting one's thinking in an axiomatic structure" (Pirie & Kieren, 1992, p. 247).

The outermost level in the model is called *inventising*, a word specially coined by Pirie and Kieren (1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994). "In inventising, a person with a fully structured knowledge breaks away from all the preconceptions which brought about this complete understanding and creates totally new questions which might grow into a totally new concept" (Pirie & Kieren, 1992, p. 247). Initially they had used the term "inventing" (Pirie & Kieren, 1989, p. 9). This term was not intended to suggest that students did not or could not develop new ideas at other levels. To avoid any misinterpretation of the fact that students can invent at other inner levels in their model, Pirie and Kieren (1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) coined the word inventising and avoided their initial use of the word inventing (Pirie & Kieren, 1989).

Figure 1 is an attempt to represent Pirie and Kieren's (1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) ideas in a two-dimensional form. It is to be noted that growth of understanding is not seen as a monodirectional process. Thus, the model is presented as a sequence of nested circles. In this way it is seen "that each layer contains all previous layers and is embedded in all succeeding layers" (Pirie & Kieren, 1994b, p. 172). Any level of understanding is itself embedded in all outer levels while, at the same time, it has embedded in it all more inner levels. Growth in understanding is then viewed as a back and forth movement among layers. Understanding is characterised as a dynamic organising process. A critical feature of this theory is that of folding back. The non-unidirectional nature of this theory is highlighted by this feature. Pirie and Kieren (1992) argued that "one can fold back to any inner level of understanding activity in order to extend one's current, inadequate understanding" (p. 248).

Growth in mathematical understanding is aided by three basic types of intervention. These are "provocative, invocative and validatings interventions" (Pirie & Kieren, 1992). From a teacher's perspective these interventions are explained in the following way:

> *Validating* intervention is intended to allow the student to show where he [or she] is and why he [or she] is thinking that way, either to himself [or herself] or an observer. A *provocative* intervention, on the other hand, is an attempt to deliberately prompt the extension of a student's understanding to new cases or to an outer level. An *invocative* intervention is intended to pose or identify an obstacle to a student's understanding and prompt folding back to inner level activities, even possibly back to directed image-making activities. (p. 248)

It is important to note here, however, that even though the teacher may have such intentions, it is the *response of the student* that actually determines the nature of the intervention, namely whether it is provocative, invocative or validating.

This model reflects its comprehensive nature by including in it as a critical element of the theory, a mode of understanding which identifies a student who is able "to operate mentally or symbolically without reference to the meanings of basic concepts or images" (Pirie & Kieren, 1992, p. 248). This feature of the model is illustrated in figure 1 by the use of bold rings. Pirie and Kieren call these rings "'don't need' boundaries" (p. 249). Once students have an image (image having level of understanding) they do not need specific examples of the preceding level, image making. Their understanding is also independent of their primitive knowing. Formalisation at the fifth level indicates that their mathematical activity does not require a concrete meaning or an image. Similarly, a student with a mathematical structure at the seventh level does not need the meaning brought to it by any of the inner levels. To be doing mathematics they do not need to perform formal algorithms and no longer need to have concrete meaning. Figure 1 indicates these "don't need" boundaries with bold circles which appear just inside image having, formalising and structuring.

An important aspect of this model that needs to be noted is the fact that not all mathematical understanding of all topics flows in a smooth connected manner even in the context of folding back. A student's understanding of a topic can be disjoint. Kieren and Pirie (1994) have termed this "disjoint understanding" (p. 51). While there is value in viewing understanding as a state that can be attained or achieved (Byers & Herscovics, 1977; Skemp, 1976, 1987), this dynamical model offers an alternative view of mathematical understanding. It is an "on-going, growing, process by which one responds to the problem of re-organising one's knowledge structures, possibly in the face of epistemological obstacles, by a process of revisiting one's existing understanding" (Kieren & Pirie, 1994). It is dynamic.

This model for growth of mathematical understanding provides a framework within which to investigate the thinking processes of the students in the current study as they attempt to make sense of the learning experiences of introductory algebra over time. It is recognised in this model that the first level, primitive knowing, is not intended to imply a low level of mathematics. There is no attempt to necessarily link better or high level mathematics with the outer levels. Using the language of levels and layers, however, Pirie and Kieren (1994b) indicated that there is some underlying hierarchy within the model.

Analysis

The relevant and significant aspects of the case study will be analysed in the first instance using this model for the growth of mathematical understanding developed by Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994). In reference to their model Pirie and Kieren (1994b) noted

> analysis can only ever be based on what the teacher observes. This notion of mapping entails plotting as points on a diagram of the model, observable understanding acts and drawing continuous or discontinuous lines between these points, dependent on whether or not the student's understanding is perceived to grow in a continuous, connected fashion. (p. 182)

An example of such a diagram is given in figure 2. It is a map of the growth of a student's understanding during an interview reported by Pirie and Kieren (1992). Since this current research will monitor the students over two years, there will be a sequence of such diagrammatic representations over time for each student in the study. Such data will contribute to the generation of a theoretical framework to explain how students construct their initial algebraic knowledge. Besides applying their model to an eight year old child, Pirie and Kieren (1994b) and Kieren & Pirie (1994) have also applied this model to university students. This indicates the broad usage of their dynamical model to trace the growth of mathematical understanding at any level in any topic.

The analysis of data in this research will inform future teaching practice in the area. However, broad generalisation will not be possible. The analysis, which is fine-grained, is contextualised within the naturalistic setting of the classroom and as such may provide insights into how students respond to the act of teaching or instruction. Significant events in the growth of algebraic understanding for each subject will be mapped over the two year period which forms the timeframe for the research.

Questions in the domain of algebra relating to observable behaviours may be used as a particular focus in mapping a particular student's growth of mathematical understanding when applying the dynamical model of Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994). If a student is able to demonstrate the ability to apply a general rule usefully within a particular mathematical topic it may suggest that for this topic this student is demonstrating the *image having* mode of understanding as a minimum.

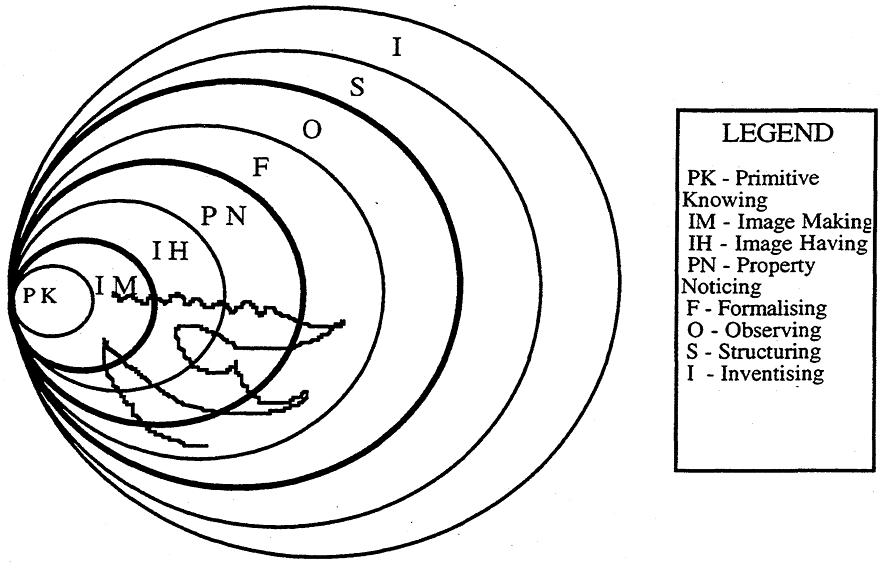

Figure 2: A map of the growth of a student's understanding during the interview. (Pirie & Kieren, 1992, p. 253).

Summary

This research aims to inform future teaching practice in the area of algebra by exploring how students construct their initial algebraic understandings of such concepts as variable and function. A case study approach is used for the study of a small group of students over two years of schooling which span the transition from primary school to secondary school. The dynamical model of Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) is used to develop the initial framework for this research. From this a theoretical framework may be generated to assist in explaining the constructions that students make in developing their initial algebraic understandings.

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