Second grader's representations and conceptual understanding of number: a longitudinal study

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This paper reports the first phase of a 2-year longitudinal study of 120 Grade 2 children, which aims to describe the growth of numerical concepts and processes and children's representations of these. Most children successfully used ten as an iterable unit but were less effective when required to formulate and identify the pertinence of groupings. Performance on ratio and splitting tasks was relatively poor, showing lack of experience in halving and sharing situations. Low achievers were more likely to produce poorly organised pictorial and iconic representations that were lacking in structure. High achievers who used dynamic imagery presented their solutions in well structured but often unconventional ways.

It is not entirely clear how children use their representations in building conceptual understanding of number. Are children's representations of numerical strategies and situations a reflection of their developing concepts, or do their representations actually enhance the development of numerical thinking? Do children make connections between critical aspects of number learning through their representations? How do children's representations change over time?

A 2-year longitudinal study aims to analyse and describe, in the greatest possible detail, children's internal representations of number and the relationships children make between various numerical processes. The study seeks to describe and explain how children impose structure, or lack thereof, on numerical situations, and how imagery influences this process. Clinical interviews focus on aspects of structure in children's strategies and representations when solving tasks involving visualising, estimating, counting, grouping, re-grouping, partitioning and multiplicative reasoning. A 2-year longitudinal study will provide a more coherent picture of how young children construct their numerical representations, and whether these representations change and develop to accommodate more complex numerical structures over time.

Background

Recent studies have focused on the importance of children's construction of number concepts and processes by examining specific areas such as counting, addition and subtraction, or multiplication and division as separate areas of study. Studies focusing on numeration have primarily examined children's counting and grouping strategies (Bednarz & Janvier, 1988; Denvir & Brown, 1986a,b; Fuson, 1990; Hiebert & Wearne, 1992). Other recent studies analysing the growth of number concepts and processes indicate that children's representations of problem-solving situations are closely linked to their conceptual understanding and the way they construct mathematical relationships (Davis, Maher, & Noddings, 1990; Goldin & Herscovics, 1991; Maher, Davis & Alston 1993; Mulligan & Mitchelmore, 1996). Researchers have also highlighted children's representations as evidence of the idiosyncratic and creative ways in which they structure mathematical relationships (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1996).

In recent years, research on multiplicative reasoning has looked to the early development of multiplication and division and fraction concepts, through essential processes such as counting, partitioning, grouping, unitising and 'splitting' (Confrey, 1994; Lamon, 1996). Studies investigating multiplication and division processes have identified the development of sound problem-solving strategies from an early age and the importance of modelling and representation in this development (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1996). Mulligan & Mitchelmore (in press) found that the strategy children employed to solve a multiplicative problem did not necessarily reflect any general problem characteristic but rather the mathematical structure that the student imposed on the problem. The children's internal conceptual structures of multiplication and division, described in terms of intuitive models, were inferred from elements such as grouping, partitioning, counting and patterning in their external representations and their explanations of these representations.

Imagery and Representations

A number of theoretical approaches are concerned with the way imagery is central to the construction of children's mathematical concepts and processes. The early work of Piaget and Inhelder (1971) referred to the importance of images as internal, holistic representations of actions which could be inspected and transformed. Other researchers have defined the mental processes involved in the child's construction of mathematical concepts as different types of image processing (Brown & Presmeg, 1993; Pirie & Keiren, 1992; Pitta & Gray, 1996; Sfard, 199; Thomas & Mulligan 1995).

Pirie and Kieren (1992) proposed a model of the growth of mathematical understanding described as a dynamic organising process involving 'image making' and 'image having'. The recent work of Pitta & Gray (1996) also draws attention to the way children use imagery to concentrate on different objects, or different aspects of the objects, which are integral components of numerical processes. They claim that the quality of the image formed from enactive approaches, is dependent upon what it is the child chooses to create an image of. This will influence the use to which the image is put. Children's interpretations of mathematical actions are so strongly associated with image formation that it is possible that children's interpretations of mathematical actions may be pre-ordained by their interpretation of the real world. Further research in the acquisition of numerical concepts may then find better explanations for children's difficulties by focusing on the imagery associated with the objects involved in concrete-abstract thought.

However, few studies have provided an in-depth analysis of children's imagery associated with essential and related features of number such as visualising, estimating, counting, grouping, regrouping and 'splitting'. Furthermore, there is growing need to examine children's construction of number concepts and strategies over time through longitudinal studies that monitor growth and change.

This 2-year longitudinal study funded by a large grant from the Australian Research Council, aims to describe the growth of children's conceptual understanding of number by examining qualitative differences in their strategies and representations of number over time. Although the study cannot provide very detailed descriptions of this 'growth' that might well be provided through case studies, it aims to describe patterns of growth and to identify relationships, based on 120 individuals' responses to a set of numerical tasks.

Research Methodology

To date, clinical interviews have been conducted with each child three times– once each in April and November of 1996, and once in April 1997, with the final interview to be conducted in November, 1997. Pilot work was conducted prior to these interviews to trial procedures, develop tasks and coding methods, and for training purposes (Mulligan, Mitchelmore, Outhred & Bobis, 1996).

Sample

The interview sample comprised 120 Grade 2 students, balanced for gender, and ranging from 6.5 to 7.5 years of age at the time of the first interview; of these 117 remained at the third interview. Twelve students were randomly selected from each of ten NSW Department of School Education primary schools randomly selected from three metropolitan regions of Sydney. The sample was representative of students from diverse cultural, linguistic and socio-economic backgrounds.

Interview Tasks

Seven categories of tasks were formulated from those trialled in pilot work that had a common set of characteristics: visualisation/ counting, ratio, grouping, regrouping and partitioning, estimation and 'splitting'. Grouping tasks focused on equivalent grouping using ten as an iterable unit, partitioning and estimating (Bednarz & Janvier, 1988; Cobb, Perlwitz & Underwood, 1996; Lamon, 1996; Mulligan & Mitchelmore, in press; Steffe, 1991). Numeration tasks focused on structural elements of the number system, visualisation and number sense (Hiebert & Wearne, 1992; Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1996). Other tasks focused on ratio and multiplicative reasoning such as arrays and 'splitting', and these were adapted from those used by

Confrey (1994), Mulligan & Mitchelmore (1996) and Outhred (1993). The tasks are shown in Table 1.

Table 1 Task-Based Interview Items

1. Numbers in Context: How old are you? What year is it? How old will you be in 2 years time? How old will you be in 5 years time?
2. Counting sequence; visualisation tasks: **2.1** Ask child to close their eyes and to imagine the numbers from 1 to 100, explain, then draw and write about what they see. Check for other sensory stimulation, e.g., movement, position, colour, or feelings associated with numbers. Repeat the activity to find what the children imagine for: **2.2** numbers past 100; **2.3** before 1; **2.4** before 0 **2.5** between 0 and 1.
3. Ratio: 3.1 You can trade 2 small stickers for a large sticker. How many large stickers are worth the same as 4 small stickers? **3.2** How many large stickers are worth the same as 2 large stickers and 4 small stickers?
4. Grouping/addition: 4.1 Show a collection of pencils in packets of 10 and as loose pencils. Ask students to show 26 pencils. **4.2** Can you show me 9 more than this? How many is that? **4.3** Ask children who are successful with above tasks if they can find another way to show 35. If child does not group; Can you arrange them to make it easier to count?
5. Regrouping, addition-ones, tens, hundreds: 5.1 Show lollies in bags, rolls & loose. Can you show me 125 lollies? **5.2** Show me 10 more than this. How many altogether now? **5.3** How would you make 100 more than this? How many altogether then ?
6. Estimation/grouping: Show a picture of 65 dots randomly drawn. Could you do something so that you will be able to tell me very quickly how many dots there are? What should you do?
7. Multiplicative: 7.1 Splitting (No materials) If you had 24 textas how would you share them fairly between 2 children (of the same age)? **7.2** If two more children came along, how would you share them fairly? **7.3 Multiple Count** Close your eyes and count in 3's as far as you can. What do you see in your mind when you count in 3's? Draw and explain. **7.4 Equal groups** There are 3 children and each child needs 7 textas. How many textas are needed altogether? (Provide textas and allow to draw). **7.5 Array** Show a 3 x 3 array of dots briefly. Cover. How many dots are there? Draw the array from memory. If unsuccessful show array and ask child to copy it. **7.6 Subgrouping** Use playdough to imagine and make a lolly snake. Show how to share 2 lolly snakes fairly between 4 children. How much do they get each?

Interview Procedures

All interviews were conducted by a trained research assistant in a small room separate from the classroom. The tasks were read to the child by the interviewer and were re-read to the child as often as necessary to assist them in remembering details. Paper and pencil were available on the table as well as the necessary materials for each task. The interviewer explained to the child that they could use the materials and pencil and paper to assist in solving the problem if they wished. The child was asked to represent their solutions by modelling, drawing and symbolising their mental images. They were also asked to explain their representation and their solution process. The interviewer asked whether the child could provide any other solutions or images. The child was given ample time to draw representations and were instructed that their drawings needed only be rough sketches.

Task 1 was designed to establish rapport with the child and identify whether the child could use concepts of age and time. The visualisation tasks (Tasks 2.1 to 2.5 in Table 1) were asked prior to other tasks so that the influence of representations used in the other tasks could be minimised. The tasks were administered for all interviews in the order shown in Table 1. The interviews lasted from 15 to 45 minutes with an average time of 30 minutes. When a child gave an explanation that appeared to conflict with the interviewer's observations, they were asked to describe, model or draw what they were thinking at that time.

Data Coding

The interviewer recorded children's responses as they solved each problem, drawing diagrams of children's modelling and noting explanations, gestures and finger movements. A coding system was developed from pilot data for each task category and responses were noted immediately in these categories. This included coding as incorrect, uncodable and non-attempt. Children's drawings were collected and coded according to the identified categories. Additional categories appeared for Task categories 2 and 7. Each interview was also audiotaped. A second trained research assistant checked the interviewer's coding, notes and audiotapes and clarified uncertainties in consultation with the chief investigator and the interviewer. In all, 19 errors were found and corrected. The coding definitions were used to check for consistency and an 89% agreement rate was found. Two research assistants also checked the data entry and the coding sheets for consistency.

Analysis of Strategies and Representations

Data from the first interview were initially analysed for performance and strategy use to identify some general trends. Additionally, children's drawings and their explanations were analysed for three characteristics:

(a) the type of representation (pictorial, ikonic or notational, including identification of colour and position);

(b) the evidence of structural development of the concepts and processes shown in the strategies or representation; and

(c) evidence of the static or dynamic nature of the image.

For example, pictorial recordings were defined as pictures drawn or oral descriptions of objects given by a child. Iconic recordings included drawings of tally marks, squares, circles or dots that represent numbers. Notational recordings were distinguished by the predominant use of numerals and symbols. The level of structural development was interpreted from structural elements (i.e. grouping, regrouping, partitioning and patterning) given in the responses and in the recordings. The static or dynamic nature of the response was defined according to whether the recordings and children's explanations of their representations describe fixed or moving (changing) entities.

Results

The present paper presents initial analysis of Interview 1 data comparing general performance and strategy use across a range of number tasks. Some exemplars of children's drawn representations will be discussed in the presentation of the paper.

Table 2 shows a general classification of children's representations for the five visualisation tasks 2.1 to 2.5. These data are presented in three main categories: pictorial recordings, notational recordings, and children who explained that they saw 'nothing in their mind'. A separate category of 'dynamic imagery' indicates the percentage of responses using imagery of number that was moving or changing or 'rolling across their mind like a screen'. Table 2 indicates that the majority of children gave notational representations for visualisation task 2.1 (76%), task 2.2 (68%) and task 2.3 (80%) respectively. Notational representations included numerals in a variety of sequences, positions and arrangements. Pictorial responses resembled drawings of animals and objects.

When asked to visualise numbers 'before 1' (task 2.3), it is noted that 70% of children visualised the numeral zero with 10% identifying other numerals. For the visualisation task 2.4 (before 0) there was a clear distinction between those children who could describe and symbolise fractions such as 1/2 and those who explained there was 'nothing', with 64% of children unable to visualise anything. Of the 30% of children who gave notational responses, 23% gave negative numbers and 10% explained that there was an 'infinite number of numbers'. Most children were unable to imagine anything for visualisation task 2.5 (between 0 and 1), but of the 30% who responded, 20% of responses indicated images such as zeros becoming larger and larger, common and decimal fractions, and negative numbers. Table 2 also shows that dynamic imagery was evident in 17% of responses for task 2.1 but diminished markedly for the other tasks. Many of the examples of dynamic imagery were produced by children who performed well and these recordings included drawings, patterns and diagrams that were highly idiosyncratic and well organised.

Table 2 Percentage of Responses by Type of Representation for Visualisation Tasks

Task	Type of Representation			
	Picture	Numeral	Nothing	Dynamic Imagery*
2.1 Visualisation 1-100	14	76	10	17
2.2 Visualisation Past 100	14	68	18	4
2.3 Visualisation Before 1	9	80	11	3
2.4 Visualisation Before 0	6	30	64	0
2.5 Visualisation Between 0 and 1	10	20	70	0

* Percentage of responses using dynamic imagery

Table 3 compares the performance and main strategy type for task categories 3 to 7. The success rate on the grouping and regrouping tasks (tasks 4 and 5) was very high, with the ratio and estimation tasks moderately more difficult. The ratio task 3.2 and 'splitting' tasks 7.1 and 7.2 were the most difficult with 70%, 58%, and 76% of students unsuccessful respectively.

Table 3 Performance by Strategy Type for Ratio, Grouping, Regrouping, Estimation and Splitting Tasks

Task	Performance		Strategies		
	% Correct	% Incorrect	No Stickers	Stickers	
3.1 Ratio 2:1	75	25	38	37	
3.2 Grouping/ Ratio	30	70	13	17	
4.1 Grouping			Tens	Ones	
	95	5	67	28	
5 Regrouping 100's,10's,1's			Ones	Multiples	
	91	9	18	73	
6. Estimation/ Grouping			Tens	Others *	Random
	80	20	42	27	11
7.1 Splitting/ Share 24/2	42	58			
7. 2 Splitting/ Reformulate	24	76			

n = 118

* Others refers to groups of twos, fives or unitary methods

The ratio task 3.1 was solved by 75% of children by visualising or modeling the task with stickers to show the relationship 'two for one'. Those children who were unsuccessful were unable to model the relationship and either added the numbers or guessed. For the more difficult ratio task 3.2, only 30% of children could successfully use 'two for one' as a unit to determine the total, with many children counting by ones the total number of stickers in an undifferentiated way. Solution strategies for grouping task 4.1 are shown as two categories 'tens' and 'ones' showing 67% of children using ten as an interable unit. However, 28% of children counted the pencils by ones even though they could identify that there were ten pencils in each packet. Similarly, in the regrouping task 5, 73% of children used ten as an iterable unit.

In contrast, the estimation/grouping task 6 distinguished children's grouping strategies in a situation where there were no pre-determined groupings. Only 43% of children used tens to group dots, with 27% of children using groupings of two or five. The other 30% of children were unable to form groups, systematic or otherwise, even though they could articulate the task requirements. The 'splitting' tasks 7.1 and 7.2 proved to be difficult for children, with less than half able to share 24 textas between 2 children by halving, dealing, counting, or use of known facts. Reformulation strategies in task 7.2 were successfully used by those children who used effective halving and dealing strategies. The results of the splitting tasks are representative of difficulties children experienced in the other multiplicative tasks with the exception of the equal grouping task which was solved correctly by the majority of children.

Discussion of Results

Preliminary analysis of the first interview data has yielded unique evidence of the diversity of children's numerical understandings and strategies and their representations. Although some children were able to form groups of ten and use ten an iterable unit, there was an over-reliance on unitary counting methods. Although the analysis is incomplete, the data showed many cases where individuals were unable to justify the pertinence of grouping in many tasks. These children also lacked a type of flexibility in their thinking; they were only able to replicate models of groups, arrays or situations which had been produced or provided by others. It appears, at this stage, that the poor performance in ratio and multiplicative tasks can be attributed not just to lack of experience in these situations, but to the primitive idea that unitary counting can be used to solve everything.

There was sufficient evidence to show that children's imagery influenced the way they visualised and solved a range of numerical tasks. Low achievers were more likely to produce poorly organised, pictorial and iconic representations that were lacking in structure. High achievers who used dynamic imagery, presented their solutions in well structured but often unconventional ways.

Most striking was the evidence of how children created and imposed structural aspects on their solutions to number tasks. In cases of low achievers, children were unable to visualise any structural characteristics and were so focused on their idiosyncratic interpretation of reality that the mathematical tasks and the concrete objects embedded in them had a completely different meaning to that intended.

On the other hand, there were some cases where the children represented mathematical structures to solve a range of tasks. This included number patterns and relationships involving ratio and proportion and competent use of negative numbers. There were children who had already developed a coherent system for dealing with numbers and these children were also able to visualise numbers in a structured way. They could produce a variety of responses to the same task, and explain that there were similarities between their responses to a number of tasks.

The critical aspect of this investigation is that children's representations reflect their conceptual understanding to some extent. However, it appears from the data so far that we need to focus more closely on why the abstraction of concepts is determined by the concrete or visualised reality of the child. There is evidence that some children are unable to move from concrete to abstract thinking, or visualise mathematical situations at all. Further analysis needs to focus on how children use imagery in the formation of concepts and why some children are unable to create or use an organised 'structure' in developing concepts.

After completion of data collection, the analysis will focus on describing the changes that occur in the development of an individual's number knowledge and how children relate visualising, counting, grouping, estimating and multiplicative processes. In particular, the research will address these questions: What are the critical aspects of imagistic thinking that influence the growth of numerical understandings? Can the difficulties faced by many students in counting, grouping, regrouping, and multiplicative reasoning be attributed to methods of instruction, or intuitive processes in the early formulation of number concepts? How can we use new information about the way imagery influences children's representations of number in the investigation of other mathematical concepts?

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