The relationship between the place value understanding of seven-year-old children and the strategies that they use to solve written addition problems

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Individual interviews were used to explore 128 seven-year-old children's understanding of place value, to see how this related to the strategies that they used to solve written addition problems. Children who understood the place value of two digit numerals were more successful at solving written addition problems, but a surprising number of children with face value understanding could also solve the problems. The variety of strategies (successful and unsuccessful) that the children used are described, and the implications for teaching algorithms are discussed.

In New Zealand, as children progress from the Junior school to the Standards (at around age seven) they are working on some key concepts such as decade structure and place value (Faire, 1992). The nature of these concepts may give rise to particular problems, especially where the written numerals and the spoken number words are two related but different systems (Carpenter, Fennema, & Franke, 1991; Kamii, 1985; Miura et al, 1993). This has implications for children learning mathematics in English (as opposed to those learning mathematics in languages like Maori, where the structure of the number system is explicit in the number words). As Carpenter et al (1991) explain, in English, while the number words larger than one hundred explicitly designate the units and the number of each unit, with numbers less than one hundred the designation of units is less explicit. The number names do not clearly emphasize the groupings of tens, and the problem is even more acute in the teens, where numbers are designated with a single word, and the first syllable of the word denotes the *units* rather than the tens (e.g. *four*teen). This is reflected in the way that many young children write the English "teen" numerals (e.g. fourteen written as 41) (see Peters, 1994), and may have implications for place value understanding. A study by Miura et al (1993) indicated that the place value understanding of children in countries like Japan and Korea, where the numerals correspond exactly to the spoken number words, was better than in countries like the United States, France and Sweden, where the number names do not clearly show the groupings of numbers.

In addition, many children seem to have problems with the formal language of mathematics and the use of symbols (Hughes, 1983, 1986; Skinner, 1990). Work with five to seven-year-old children showed that the children were reluctant to use the operator signs to represent addition and subtraction, even though these symbols were being used in their maths books (Hughes, 1983, 1986). It has been suggested that a major difference between high maths aptitude children and those who are poor mathematics learners, is the extent to which they are able to make sense of the rules and symbols they are taught (Resnick, 1986).

If children are involved in using symbols that they do not understand, they may conclude that mathematics is not supposed to make sense. If this happens, they are likely to stop monitoring their work thoughtfully and will not be troubled by answers that are clearly wrong (Baroody & Ginsburg, 1990). Instead of focussing on the mathematics they may adopt a passive kind of involvement in the classroom and learn the advantages of self preserving behaviours (King, 1993). These may include adopting systematic routines, even if they yield wrong answers (Brown & vanLehn, 1982; Resnick & Omanson, 1987). For example, strategies which children employ to cope with story problems that they do not understand may include; guessing, trying all the operations and looking for the most reasonable answer, and looking for isolated key words and phrases that signal operations (Sowder and Sowder, 1989).

If the children resort to these 'coping strategies' rather than coming to understand the mathematical concepts that are involved, it may help to explain why the gap between children at the top, and those at the bottom, widens as children progress through school (see Fogelman & Goldstein, 1976; Young-Loveridge, 1991).

The purposes of this study were, firstly, to explore children's understanding of the place value nature of written numerals and secondly, to note how this related to the strategies that they use to solve written addition problems.

Method

This study involved 128 seven-year-old children from three New Zealand primary schools. All of the children were in their third year of school (J3). The children were from a variety of ethnic backgrounds but all had English as their first language and were being taught in English.

Individual task based interviews were used to explore the children's understanding of place value and the strategies that they used to solve written addition problems.

The place value task used in this study was adapted from a task used by Faire (1992) and Kamii (1985), and was designed to assess the children's understanding of the place value nature of a two-digit numeral. (For full details of the task see Peters, 1994.)

Figure 1 shows the four written addition problems that were used. The researcher recorded the strategy that the children used to solve these problems, and the direction in which they worked (e.g. from right-to-left, adding the units first, or from left-to-right, adding the largest numbers first).

37	75	84	325
+21	+15	+39	+285

Figure 1. Written addition problems

Results and Discussion

Place value understanding

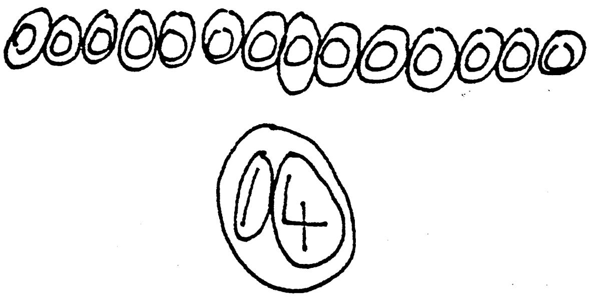
Data from the place value task were analyzed, using a series of levels of place value understanding (described below), which were adapted from those described by Faire (1992, pp45-46), Kamii (1985, pp56-57) and Ross (1989, pp49-50). Copies of the children's drawings illustrating the different levels of place value understanding are shown in Figure 2.

Level 1a, Cardinal Representation: Here objects are counted out to represent numbers. That is, the set of counted objects is the meaning of the number. No meaning is assigned to the individual digits.

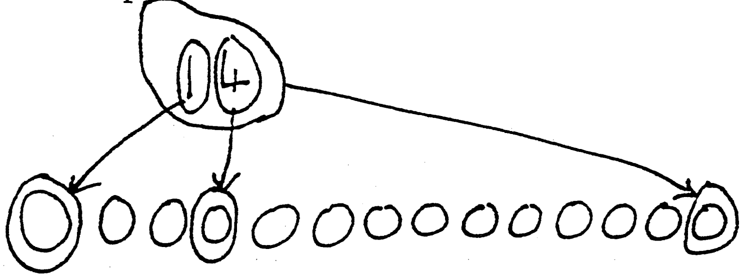
Level 1b, Ordinal Representation: The individual numerals represent ordinal positions. For example, the one in 14 represents the first counter in a line of 14 counters, the numeral 4 represents the fourth counter, and the whole numeral represents the fourteenth counter.

Level 2, Face Value Representation: Children at this level recognize the face value of each of the digits and that both digits taken together represent the whole set of objects, but do not recognize that the number represented by the tens digit is a multiple of ten. For example, the one in 14 represents one counter, the four in 14 represents four counters, although the whole numeral represents 14 counters. The fact that nine counters remain unaccounted for is of no concern to these children

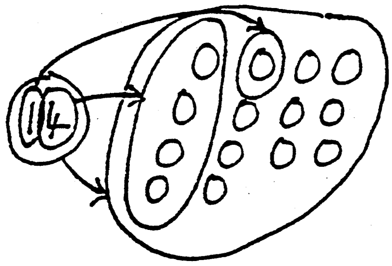
Level 3, Positional Understanding: The individual digits making up a two digit numeral stand for the amounts that are determined by the place or position in which the digits occur. The tens digit is interpreted as a multiple of ten and the right hand digit as ones. The total of tens and ones is understood as the cardinal number of the set of objects.



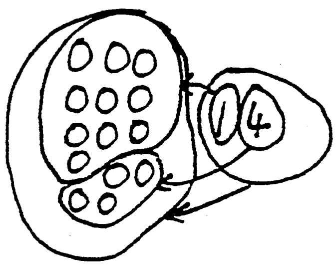
Level 1a, Cardinal Representation



Level 1b, Ordinal Representation



Level 2, Face Value Representation



Level 3, Positional Understanding

Figure 2. Examples of children's drawings showing different levels of place value understanding

Table 1 shows the number and percentage of children at each level. The majority of the children were at Level 2, they recognized the face value of the digits and that both digits taken together represented the whole set of objects, but did not recognize that the number represented by the tens digit was a multiple of ten. A small number of children had not yet developed any understanding of place value and were not able to show that the cardinal value of a set of objects can be represented by a multi-digit number. These results were consistent with previous findings, which indicate that some children are still constructing meaning for place value after several years of schooling (Faire, 1992; Kamii, 1985; Ross, 1989; Seirink & Watson, 1991).

Table 1
Number (and percentage) of children at each of the levels of place value understanding

	N	(%)
No understanding	9	(7)
Level 1a	9	(7)
Level 1b	8	(6)
Level 2	75	(59)
Level 3	27	(21)

Some of the topics that were covered during the mathematics lessons observed in this study, such as two-place addition and subtraction with renaming, require an understanding of place value if they are to be fully understood. As these results show, very few children had reached a full understanding of place value when they started to work on these topics. Although it was not discussed, some of the teachers involved in this study may have over-estimated the children's ability to understand the place-value nature of two-digit numerals. Ross (1989) suggests that it is easy for teachers to over-estimate children's understanding in this area because children may appear to understand more than they do. When faced with a collection that is already grouped in standard place value partitioning of tens and ones, a student who is asked to make correspondence for the digits in 52 for example, need only look for "five of something and two of something else" (p50).

The fact that such a small proportion of the children in this study interpreted the tens digit as a multiple of ten had implications for their ability to do written addition which required regrouping and renaming. The following section looks at the results of the written addition tasks, and the relationship between place value understanding and the strategy used to solve written addition problems.

Strategies used to solve written addition problems

More than half of the children (61%) were able to complete 37+21, a simple addition problem which does not require regrouping. Thirty-eight percent were able to solve 75+15. This problem does not involve renaming the tens, and the addend was small enough for some children to solve this using a counting-on strategy. Only 23% were able to solve 84+39, which required regrouping and renaming. Although very few children had been taught 3-place addition, a similar number (20%) were able to solve 325+285. It appeared that once the principles of regrouping were understood, this knowledge transferred to problems involving larger numbers.

Table 2
Number (and percentage) of children at each of the levels of place value understanding who were successful on each of the written addition problems.

	37+21		75+15		84+39		325+285	
	N	(%)	N	(%)	N	(%)	N	(%)
No understanding	2	(3)	0	(0)	0	(0)	0	(0)
Level 1a	6	(8)	2	(4)	1	(3)	1	(4)
Level 1b	3	(4)	3	(6)	3	(10)	1	(4)
Level 2	44	(56)	23	(47)	13	(43)	10	(40)
Level 3	23	(30)	21	(43)	13	(43)	13	(52)
Total	78		49		30		25	

Table 2 shows that most of the children who were successful on the written addition problems were either at Level 2 or Level 3 of place value understanding. A surprising number of children who only recognized the face value of digits (Level 2) were able to solve these problems. Figure 3 shows a strategy which was commonly used by Level 2 children, where the face values of the digits were added and the total written below the column. This strategy resulted in a correct answer for 37+21, but the children needed to carry out some form of regrouping to be successful on the other three problems. Figure 4 shows an example of a Level 2 child using a traditional carrying procedure to successfully solve all of the problems. The fact that many Level 2 children used the procedure shown in Figure 4 indicates that children were learning to apply this 'carrying' procedure before the place value nature of larger numerals was fully understood. Other authors (e.g. Resnick & Omanson, 1987) have also found a lack of association between ability to perform the 'carry' procedure, and an understanding of the number system.

3 7	7 5	8 4	3 2 5
+ 2 1	+ 1 5	+ 3 9	+ 2 8 5
5 8	8 10	11 13	5 10 10

Figure 3. Strategy commonly used by Level 2 children to solve written addition problems (adding the face value of the digits).

	1	1	1 1
3 7	7 5	8 4	3 2 5
+ 2 1	+ 1 5	+ 3 9	+ 2 8 5
5 8	9 0	12 3	6 1 0

Figure 4. Level 2 child using a traditional 'carrying' procedure.

Although many children were able to correctly use the 'carrying' procedure for solving written addition problems, many others showed that they were trying (unsuccessfully) to apply a procedure that was not fully understood. Figures 5, 6. 7, and 8 show some examples of common errors. Aaron (see Figure 5) was unsure where to put the carried amount, and, when he did carry an amount to the next column, he did not include it in his total. Terri (see Figure 6) attempted to carry the 0 (ones) when adding 75+15, instead of the 1 (ten).

	1	1	1 1
3 7	7 5	8 4	3 2 5
+ 2 1	+ 1 5	+ 3 9	+ 2 8 5
5 8	8 0	1 2	5 0 0

Figure 5. Aaron: Example of an incorrect carrying procedure.

$$\begin{array}{r} 3\ 7 \\ +\ 2\ 1 \\ \hline 5\ 8 \end{array} \qquad \begin{array}{r} 7\ 5 \\ +\ 1\ 5 \\ \hline 8\ 1 \end{array} \qquad \begin{array}{r} 8\ 4 \\ +\ 3\ 9 \\ \hline 11\ 13 \end{array} \qquad \begin{array}{r} 3\ 2\ 5 \\ +\ 2\ 8\ 5 \\ \hline \end{array}$$

Figure 6. Terri: Example where 5 and 5 were added (getting the answer 10), the child carried the 0 (ones) instead of 1 (ten)

When children worked from left-to-right, instead of from right-to-left, regrouping or 'carrying' became even more problematic. When solving 75+15, Jo (see Figure 7) added the tens column first, writing 8 below the column. He then added the ones, getting the answer 10. He carried 1 across to the top of the tens column, and wrote 9 (ten minus the carried 'one') in the answer line. The carried amount was not incorporated into his answer.

$$\begin{array}{r} \\ 3\ 7 \\ +\ 2\ 1 \\ \hline 5\ 8 \end{array} \qquad \begin{array}{r} 1 \\ 7\ 5 \\ +\ 1\ 5 \\ \hline 8\ 9 \end{array} \qquad \begin{array}{r} \\ 8\ 4 \\ +\ 3\ 9 \\ \hline \end{array} \qquad \begin{array}{r} 1 \\ 3\ 2\ 5 \\ +\ 2\ 8\ 5 \\ \hline 5\ 1\ 10 \end{array}$$

Figure 7. Jo: Example where 5 and 5 were added (getting the answer 10), the child carried 1 and wrote 9 (10-1) in the answer line.

Some children who worked left-to-right carried an amount to the previous column, but because their answer was already in place, they did not incorporate it into their answer. Rene's answers provide an example of this (see Figure 8). When solving 75+15 she added 7 and 1 to get 8, and wrote 8 beneath the tens column, she then added 5+5, getting the answer 10. She wrote 0 in the ones column, and put the 1 above the tens column, but did not incorporate this into her answer. She used a similar strategy on the other two tasks (84+39 and 325+285).

$$\begin{array}{r} \\ 3\ 7 \\ +\ 2\ 1 \\ \hline 5\ 8 \end{array} \qquad \begin{array}{r} 1 \\ 7\ 5 \\ +\ 1\ 5 \\ \hline 8\ 0 \end{array} \qquad \begin{array}{r} 1 \\ 8\ 4 \\ +\ 3\ 9 \\ \hline 11\ 3 \end{array} \qquad \begin{array}{r} 1 \\ 3\ 2\ 5 \\ +\ 2\ 8\ 5 \\ \hline 5\ 10\ 0 \end{array}$$

Figure 8. Rene: Example showing addition left-to-right working where carried numbers are not incorporated into the answer.

The results showed that many children were using systematic, but wrong strategies. Previous research has identified similar errors when children attempt to use the standard algorithms (Brown & vanLehn, 1982; Ginsburg, 1981; Kamii, 1985; Resnick & Omanson, 1987). This is clearly an example of when the formal mathematics of school fails to mesh with the children's informal understanding of mathematics. Perhaps because of a lack of place value understanding of the numerals, children were failing to check if their answers were realistic (for example, Aaron's 84+39=12 shown in Figure 4).

Directional strategies

Although all of the teachers taught children to work from right-to-left when solving this type of problem, only 39% of the children worked in this way. Thirty-eight percent of the children did not have a clear directional strategy and the remaining 23% tackled the addition problems working from left to right. Although working in the traditional right-to-left (<-) direction was the most successful strategy, many children who did not work in this way were also able to complete these tasks. Table 3 shows the number and percentage of the children using each directional strategy who were able to solve each of the addition problems.

Most of the children who did not have a clear directional strategy were unable to solve the problems, although five children were able to solve the first two problems by counting-on in ones. It was interesting that many children were able to successfully work from left-to-right (i.e. adding the largest numbers first). Some of the most able children worked in this way, holding the numbers in their heads, and then writing the final answer. Other children wrote the totals beneath each column, adjusting the previous figure if necessary. An example of this was Simon. On the second problem (75+15) he added the tens column (7+1=8) first and wrote 8 beneath the tens column. He then added the ones (5+5=10), wrote 0 in the ones column, and added one to the tens column, changing his 8 to a nine. He used the same strategy on the hundreds column of the last problem (325+285). These children appeared to have developed their own, meaningful strategies for doing this type of problem.

Table 3

The number (and percentage) of children using each of the directional strategies who were successfully able to complete each of the written addition problems.

	No directional strat.		Left to right (->)		Right to left (<-)	
	N	(%)	N	(%)	N	(%)
Total	48		30		50	
37+21	5	(10)	25	(83)	48	(96)
75+15	5	(10)	10	(33)	34	(68)
84+39	0	(0)	4	(13)	26	(52)
325+285	0	(0)	8	(27)	17	(34)

Sowder and Schappelle (1994) also found many children successfully using a left to right strategy, and suggest that working in this way reinforces place value understanding and is less likely to lead to errors than standardized procedures which have been memorized but poorly understood.

It is important to remember that choice of algorithms is arbitrary. There is a tendency to feel that the standard procedure selected by a particular culture is the only "right" way, and yet people from different cultures, and from different periods in history, often use very different algorithms (see Philipp, 1996). Awareness of the range of alternative algorithms which are in use around the world may make teachers feel more comfortable in accepting the alternative strategies they see children using.

Conclusion

The results of the present study are consistent with the literature that stresses the importance of place value understanding (Faire, 1990; 1992; Miura et al, 1993; Ross, 1989). Less than a quarter of the 128 J3 children interviewed in this study could show that the tens digit was a multiple of ten. This is consistent with previous findings, which indicate that some children are still constructing meaning for place value after several years of schooling (Kamii, 1985, Ross, 1989; Seirink & Watson, 1991). The fact that most of the children were being taught the conventional algorithms for addition, including the 'carrying' procedure suggests that the teachers may have overestimated the extent to which the children in their classes fully understood the place value nature of numbers.

Ross (1989) suggests that the nature of standard place value tasks makes it easy for children to appear to know more than they actually do. Teaching children conventional algorithms before they fully understand the place value nature of the number system may have consequences for their long-term progress in mathematics. Several authors have found that some children develop systematic errors when attempting to use the standard algorithms (see Brown & vanLehn, 1982; Ginsburg, 1981; Kamii, 1985; Resnick & Omanson, 1987), and Baroody & Ginsburg (1990) suggest that there is a danger that such children will conclude that mathematics is not supposed to make sense. When that happens they are likely to stop monitoring their work carefully, and will not be troubled by answers that are clearly wrong.

Many authors have described activities which could be useful in enhancing children's place value understanding (e.g. Faire, 1992; Sowder & Schappelle, 1994; Ross, 1989). It may also be important for teachers to allow children to have lots of experience working with large numbers and solving meaningful problems, using their own informal strategies, before the conventional algorithms are introduced. Where children develop their own successful algorithms teachers may need to consider what advantages conventional algorithms offer these children. It may better to allow children to use their own algorithms, which focus on number meaning, rather than rote learning conventional routines which may be poorly understood.

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