

Mathematical Misconceptions -- Can We Eliminate Them?

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If students are to successfully tackle tertiary mathematics, one prerequisite is the mastery of a number of basic concepts. Despite the best efforts of teachers, many students develop mathematical misconceptions. Is it possible to eliminate these misconceptions? In this study, a strategy based on Piaget's notion of cognitive conflict was employed for this purpose. The method used was found to be successful in reducing, and in some cases totally eliminating, the frequency of misconceptions exhibited by a class of students.

Introduction

Due to the sequential nature of many branches of mathematics, it is essential that individuals attempting to learn mathematics at a particular level first master the fundamental mathematical concepts and skills which are pre-requisite knowledge for success at that higher level (Swedosh, 1996). Swedosh asserted that mastery of this pre-requisite knowledge depends very heavily on individuals having a thorough understanding of a limited number of basic concepts and having practised certain skills to the point where they can confidently use them. It was also asserted that the understanding of these concepts, as well as the ability to use this knowledge, is so crucial to subsequent learning of mathematics, that it is of great interest to mathematics educators (teachers, lecturers, and tutors) to be able to ascertain information about the types of misconceptions commonly held by students and the frequency with which these misconceptions occur.

An understanding of the basic mathematical concepts which should be acquired in secondary school is crucial to the preparedness of students to undertake tertiary mathematics subjects (Swedosh, 1996). This issue is important from the viewpoints of prospective tertiary mathematics students and of the Victorian Government whose stated policy is that it is committed to increasing participation in post-secondary mathematics education (Victorian Government, 1987) and that educational programs in Years P--12 should provide all students with a sound preparation for further schooling (Ministry of Education Victoria, 1984).

The importance of this issue is also recognised in A National Statement on Mathematics for Australian Schools (Australian Education Council, 1990). The National Statement includes as one of the goals for mathematics in Australian schools the aim that "as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively" (p. 18).

The significance of this goal is highlighted when one considers that "contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies" (Blyth & Calegari, 1985, p. 312). Also, "statistics collected by the Mathematical Association of Victoria (MAV) show that 75% of all tertiary courses require a pass in HSC mathematics" (Blyth & Calegari, 1985, p. 312).

There are a large number of types of mathematical misconceptions, "and a complete list may not even be practical" (Davis, 1984, p. 335). Many authors have written about how a consideration of the mathematical misconceptions which pupils exhibit can be utilised to develop teaching techniques which are directed towards the diagnosis and elimination of these misconceptions (Bell, 1982; Farrell, 1992; Margulies, 1993; Perso, 1992).

It is vital to have a strategy to deal with misconceptions when they do occur. An approach which has been successful in helping many students overcome misconceptions is the "conflict teaching approach", based on Piaget's notion of cognitive conflict. In this

approach, teachers discuss with the learners (students) the inconsistencies in the thinking of the learners in order to have the learners realise that their conceptions were inadequate and in need of modification (Tirosh, 1990). Vinner (1990) supports this approach and states that "there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way" (p. 97).

A number of authors (Stavy & Berkovitz, 1980; Strauss, 1972; Swan, 1983) have found the conflict teaching approach to be an effective way of removing and correcting a variety of misconceptions involving aspects of mathematics and physics. It should be noted that there is no attempt in any of the studies reported in the papers by these authors to eliminate misconceptions in the topics considered in our paper, and the subjects being tested by us constitute a vastly different cohort.

In an earlier study, Swedosh (1996) presented information about the nature of mathematical misconceptions which were commonly exhibited by mathematics students at the University of Melbourne (U. of M.) and at LaTrobe University (LaTrobe) upon entry to a tertiary mathematics subject. A list of these misconceptions and their frequencies of occurrence was provided by Swedosh, together with a discussion of why they may have occurred.

In the study reported on in this paper, an experiment to help students overcome their mathematical misconceptions was devised so that the success or otherwise of the strategy outlined above could be tested.

Methodology

The students in a first year lecture group in Applied Mathematics at the U. of M. were given a short test. The test comprised a number of questions similar to those previously posed in the earlier tests at the U. of M. and at LaTrobe (Worley, 1993) and which yielded a high frequency of misconceptions (Swedosh, 1996) as well as some questions which were similar to those which appeared on the list of misconceptions provided in 'Algebraic Atrocities' (Margulies, 1993, p. 41).

The rationale adopted here was to provide students with the opportunity of exhibiting the misconception, if they do in fact possess that misconception. In other words, if the student does have particular misconceptions, the questions posed on the short test are intended to invite responses which exhibit this fact.

After the test had been administered, each question on the test paper of every student was examined carefully to gain information on the misconceptions exhibited. The occurrence of each misconception on each question was recorded (as well as the occurrence of those who attempted the question and who answered correctly) and the frequencies of these occurrences were then compared from question to question. Those questions on which the greatest number of misconceptions had been exhibited then became the focus of this study (the 'focus' questions were those where the frequency of misconceptions exceeded 6%).

After these 'focus' questions had been determined, time was spent in class, using the 'conflict teaching approach', attempting to eliminate these misconceptions and replace them with the correct concept. For example, if a large number of students were to exhibit the misconception that if $\frac{1}{x} - \frac{1}{b} = \frac{1}{a}$, then $x - b = a$ and so $x = a + b$, a suitable approach would be the following: Students would be shown an equation which they knew to be self-evident such as $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. They would then be shown that if one applied the same method as above, it would yield the absurd result that $2 - 3 = 6$. Having (hopefully) eliminated (or at least strongly challenged) the misconception, the correct concept would then be taught (use of a common denominator in the example above).

A new test was assembled which comprised only the 'focus' questions. This test was administered to the same class three weeks after the teaching session described in the previous paragraph. The reason for waiting three weeks after that session is that it is desirable that students are answering the questions on the new test based on what they

now believe to be the relevant concepts, not based on the memory of what they had recently heard. Students did not know that they would be asked to do this test.

The frequency of occurrence of misconceptions on each of the 'focus' questions was then calculated, and these were compared with the frequencies which were exhibited on the same questions before the teaching session.

If an improvement was found (less misconceptions exhibited), it was considered that it would then be interesting to determine whether this improvement was still evident if short questions were not used, but the need for the use of these concepts was embedded in some longer and less transparent question. This may be the subject of further investigation.

The Sample

The students who were tested were all enrolled in an Applied Mathematics subject in Semester 2, 1996 at the U. of M., having succeeded in a fairly demanding prerequisite mathematics subject in Semester 1 (only about 25% of first year students were permitted to attempt this first semester subject). A total of 103 students sat the first test and 108 students sat the second test. In order to be able to make a meaningful comparison of the results and ascertain whether the intervention of the teaching approach had caused a reduction in the frequency of misconceptions, a sample was selected from these students. It was thought to be important to consider the backgrounds of the students so that the results of this study could be put into context and not be over-generalised. That is, it was important that, if the teaching method was found to be successful with this group, it could be asserted that the method was helpful with these types of students -- an assertion about other types of students might be less appropriate. The students reported on in this study are those students who sat both the pre-test and the post-test, and who had completed Specialist Mathematics 3/4 (SM) as part of their Victorian Certificate of Education (V.C.E.). There were 60 students in this category.

In each of the three Common Assessment Tasks (CATs) in a V.C.E. subject, students are awarded a grade from E to A+ corresponding to a ten point scale from one to ten. Using this scale, the 60 students herein had an average mark for SM CAT 1 (the challenging problem) of 9.38, an average mark for SM CAT 2 (facts and skills) of 8.52, and an average mark for SM CAT 3 (the analysis task) of 8.50. They had an average Tertiary Entrance Ranking (a percentile with the highest possible ranking of 99.95 and with about 23 students for each .05 -- .05% of students in the state is about 23) of 93.56. Those excluded from this study include mature age students as well as interstate and overseas students whose backgrounds were quite different to the 60 students considered herein.

The pre-test

The pre-test consisted of 53 short answer questions. As previously stated, each question was designed to 'invite' the display of a particular misconception, should that misconception exist. Students were given sufficient time for nearly all of them to complete the test (about 30 minutes). An analysis of each question on the submission of each student was undertaken and a count was taken of how many students had the correct answer, how many had exhibited the misconception, how many had another wrong answer, and how many had not attempted the question.

There were 17 questions for which the frequency of misconceptions exceeded 6% and these became the 'focus' questions. These 17 questions are shown below, and the most common misconception(s) are shown to the right of each question :

Simplify expressions 1-5 as fully as possible.

Note that some expressions may not be able to be simplified. If this is the case, simply rewrite the expression in the space provided for the answer.

1. $\frac{100!}{98!}$ 2!
2. $\left(\frac{1}{2}\right)^{-3}$ $\frac{1}{8}$
3. $3^x \times 3^x$ $9^{2x}; 3^{x^2}$
4. $2^x + 2^x$ 4^x
5. $a^b \times a^c$ a^{bc}

Solve equations 6-12 for x :

6. $x^2 = 81$ $x = 9$
7. $x^2 - 4x = 0$ $x = 4$
8. $(x-1)(x^2 - 3x) = 0$ $x = 1; x = 1, 3$
9. $x^2 = x$ $x = 1$
10. $\frac{x^2 - 1}{x - 1} = 0$ $x = \pm 1; x = 1$
11. $\frac{1}{x} - \frac{1}{b} = \frac{1}{a}$ $x = a + b$
12. $\log_e x = \log_e t - t + c, c \text{ constant}$ $x = t - e^t + e^c$
13. Solve for x : $2x + 4 < 5x + 10$ $x < -2; x > 2$
14. Factorise $(2x + y)^2 - x^2$ $3x^2 + 4xy + y^2$
15. Given that $\sin \frac{\pi}{6} = \frac{1}{2}$, evaluate $\sin \frac{7\pi}{6}$ $\frac{1}{2}$
16. Given that $\sin A = \frac{3}{5}$ and that $\frac{\pi}{2} < A < \pi, \frac{4}{5}$
evaluate $\cos A$.

For question 17, indicate whether the statement is True or False. (circle one)

17. $k\left(50 - \frac{x}{5}\right)(80 - 2x) = k(250 - x)(40 - x)$ T / F
True

Teaching method

The teaching took place two weeks after the initial test. At this time the 'focus' questions had been determined and the teaching method was specifically targetted at those questions (which had the highest frequency of misconceptions). The method essentially involved showing examples for which the misconception could be seen to lead to a ridiculous conclusion, and, having established a conflict in the minds of the students, the correct concept was taught. When teaching the correct concepts, slightly different examples to the 'focus' questions were used (different numbers, etc.) to ensure that students were not able to simply remember what had been shown to them, but had to use the concept correctly. Some examples of what was taught are shown below. In many cases numerical examples were used to point out the inconsistency within the misconception.

$$\frac{4!}{2!} = \frac{24}{2} = 12, \quad (4-2)! = 2! = 2; \quad \frac{4!}{2!} \neq (4-2)!$$

$$3^2 \times 3^2 = 81; \quad 3^2 \times 3^2 \neq 9^4 = 81^2$$

$$2^3 + 2^3 = 16; \quad 2^3 + 2^3 \neq 4^3 = 64$$

$$x^2 = 9, \quad x^2 - 9 = 0, \quad (x-3)(x+3) = 0, \quad x = \pm 3$$

$$\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x+2 \quad \text{iff } x \neq 2; \quad \frac{0}{0} \text{ does not exist.}$$

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}; \quad 2 - 3 \neq 6$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}; \quad \sin \frac{4\pi}{3} = \sin \left(\pi + \frac{\pi}{3} \right) = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2} \quad \text{as } \frac{4\pi}{3} \text{ is in the third quadrant.}$$

$$k \left(10 - \frac{x}{2} \right) (200 - 5x) = k(20 - x)(40 - x), \quad (\text{True or False?})$$

$$k \left(\frac{1}{2} \right) (20 - x)(5)(40 - x) = k(20 - x)(40 - x); \quad \frac{5k}{2} = k \quad \text{which is only true for } k = 0.$$

The post-test

The post-test was given to students three weeks after the teaching session and students did not know that there was to be a second test. The post-test consisted of the 17 short answer questions shown in this paper.

Results

The tables below summarise the responses given by the 60 students to the 17 questions shown earlier. Table 1 shows the frequencies of each response given before and after teaching session. Table 2 shows the proportions of each response before and after teaching session for those who attempted the question. In each table, 'Q' is the question number.

Table 1

Q	Correct		Misconception		Other wrong		No attempt	
	Before	After	Before	After	Before	After	Before	After
1	52	58	5	0	2	2	1	0
2	56	60	4	0	0	0	0	0
3	54	56	6	2	0	1	0	1
4	39	47	8	2	12	9	1	2
5	52	57	7	2	1	0	0	1
6	49	60	11	0	0	0	0	0
7	53	59	5	0	2	1	0	0
8	53	57	6	1	1	2	0	0
9	28	57	30	2	2	1	0	0
10	46	44	12	9	2	6	0	1
11	56	57	4	2	0	0	0	1
12	36	54	15	2	4	2	5	2
13	45	50	7	4	4	6	4	0
14	41	55	7	0	10	2	2	3
15	51	51	7	6	2	2	0	1
16	27	50	23	5	4	4	6	1
17	44	54	10	4	0	0	6	2

Table 2

Q	Of those who attempted the question					
	% Correct		% Misconception		% Other error	
	Before	After	Before	After	Before	After
1	88.1	96.7	8.5	0.0	3.4	3.3
2	93.3	100.0	6.7	0.0	0.0	0.0
3	90.0	94.9	10.0	3.4	0.0	1.7
4	66.1	81.0	13.6	3.4	20.3	15.5
5	86.7	96.6	11.7	3.4	1.7	0.0
6	81.7	100.0	18.3	0.0	0.0	0.0
7	88.3	98.3	8.3	0.0	3.3	1.7
8	88.3	95.0	10.0	1.7	1.7	3.3
9	46.7	95.0	50.0	3.3	3.3	1.7
10	76.7	74.6	20.0	15.3	3.3	10.2
11	93.3	96.6	6.7	3.4	0.0	0.0
12	65.5	93.1	27.3	3.4	7.3	3.4
13	80.4	83.3	12.5	6.7	7.1	10.0
14	70.7	96.5	12.1	0.0	17.2	3.5
15	85.0	86.4	11.7	10.2	3.3	3.4
16	50.0	84.7	42.6	8.5	7.4	6.8
17	81.5	93.1	18.5	6.9	0.0	0.0

It can be seen from Table 2 that for every question there was a decrease in the frequency of misconceptions exhibited. Most of these decreases were substantial, but two of them (questions 10 and 15) were slight. In analysing the results, consideration was given to why responses to questions 10 and 15 had not shown the same level of improvement as all other questions. The authors believe that it is possible that question 6 impacted on question 10. Question 6 required students to give the two solutions $x = \pm 9$ to the equation $x^2 = 81$. The misconception that $x = 9$ dropped from 18.3% before the teaching session to 0.0% after. It is likely that, having learned this concept well, students then used it when solving $\frac{x^2 - 1}{x - 1} = 0$ without realising that if $x = 1$, they would be dividing by zero.

The reason for the responses to question 15 not improving substantially are less clear. It is possible that some misconceptions are harder to eliminate than others, or that students' basic trigonometry is not as solid as other parts of their mathematics, or that this method is not as well suited or as effective for some areas as it is for others. Questions 15 and 16 are quite similar, yet the results for question 16 improved dramatically. Perhaps students did not recognise which quadrant the angle was in.

Limitations of the Study

The major dilemma with any study such as this one is the extent to which the results can be generalised. Questions as to whether the sample was representative of the group to which a teaching method might be applied are important, as some methods are likely to vary in their effectiveness depending on the make up of the target group.

The students considered in this paper were very bright and their mathematics was, overall, very good. It is not certain that the results discussed herein would be as dramatic with students whose backgrounds were not as good. That is, it may be that the high level of ability of the students considered in this paper, meant that they were able to readily see the inconsistencies in their previous thinking, and learn the correct concepts, and that lesser students may not be able to undergo this process as readily.

It is possible that, if students were tested again at some time in the future, some of them may have reverted to their (often long-held) misconceptions. It is also possible that many of these misconceptions may occur again if the concept being considered was embedded in some longer and less transparent question.

Conclusions

The teaching method used was extremely effective and the improvement in the results greatly exceeded the expectations of the authors. It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated.

It is not clear whether the results would be as dramatic with less able students and particularly less able mathematics students or if the concepts being tested were embedded in more complex questions, but within the limitations of this study, the strategy used to eliminate mathematical misconceptions has been a remarkable success.

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