Generalising from and transferring between algebraic representation systems: Characteristics that support these processes.

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In response to misconceptions students hold with understanding the concept of a variable new teaching approaches have been introduced into many Australian Schools. These approaches entail generalising from visual patterns and tables of data. While recent research has reported many of the difficulties students experience with these new approaches, it has failed to delineate why these difficulties are occurring. It seems that these approaches call on an array of reasoning processes that, in the past, have not been considered as important to the algebraic domain. This paper delineates some of these processes and pinpoints particular characteristics of successful students.

Extensive studies such as The Second International Mathematics Study - The Fourth Assessment of Mathematics in the U.S.A. (1982) reported the misconceptions many students hold not only with understanding the concept of a variable but also in solving algebraic equations, and translating word problems into algebraic symbols. In Australia, one response to these concerns was to introduce the algebraic ideas using new teaching approaches (Pegg & Redden, 1990; Quinlan, Low, Sawyer, & White, 1987). For these approaches students are required to view the variable as a means for expressing the generalisations discovered by looking at tables of data and patterns represented by concrete or pictorial representations. Research with regard to these approaches is only beginning to appear in the literature.

Yerushalmy and Shterenberg (1994) suggested that the links between patterns and algebra are not trivial. Recent research has delineated some of the difficulties students are experiencing with these approaches (MacGregor & Stacey, 1993; Orton & Orton, 1994; Redden, 1994). Most students seemed to find it easy to describe the connection between two adjacent terms in a sequence but had difficulty describing the rule that created the sequence (MacGregor & Stacey, 1993; Yerushalmy & Shterenberg, 1994; Orton & Orton, 1994). Most did not naturally choose to express their generalisations in algebraic symbols (Kaput, 1992; Yershalmy & Shterenberg, 1994). The reflex of checking the formula against the given data was not present (Lee & Wheeler, 1989). Arithmetical incompetence prevented progress for some (Orton & Orton, 1994). The role of language in reaching generalisation has also been investigated. Stacey and MacGregor (1995) reported that the correct verbal descriptions were more likely to lead to the correct algebraic rules. Kaput (1992) suggested that over-learned natural language can in fact inhibit appropriate algebraic symbolism. Thus the focus of recent research seems to be identifying difficulties and not so much on delineating why these difficulties are occurring or identifying specific student characteristics that support these approaches.

One of the key processes required for success with these approaches is the ability to generalise. Dreyfus (1991) distinguished two different generalising processes. The first requires one to generalise from the specific to the general (e.g., generalising from tables) and the second requires transfer from one representation to the other (e.g., table to visual patterns to variable), a process requiring both synthesising and elaborating. If the aim of utilising a variety of approaches for introducing the concept of a variable is to enrich understanding of the variable then students not only need to establish links between each representation and the concept of a variable but also to be able to successfully transfer from one representation to another (Lesh, Post, & Behr, 1987). Not only do students need to successfully translate between the representations, they also need to envisage the dynamic nature of both representations and abstract the linear function represented, that is, envisage $ax + c$ as an expression in its own right. Sfard (1991, 1994) referred to these dual roles as structural and operational.

The structural concepts are 'static, instantaneous and integrative," whereas the operations is 'dynamic, sequential and detailed." (Sfard, 1991). Sfard (1991, p.18) believed the student passes from an operational conception to a structural conception by progressing through three hierarchical phases namely, *interiorisation, condensation,* and *reification.* Firstly, the learner becomes acquainted with the processes which will eventually give rise to the new concept. Secondly, lengthy sequences of operations are compacted into more manageable units. The student becomes more capable of thinking about a given process as a whole. There emerges a growing facility with alternating between different representations. Thirdly, the ability to see something familiar in a totally new light evolves; one has abstracted the concept.

While Sfard's (1991) theory provides considerable insights into the processes of algebraic abstraction of a concept it does not explicate the means by which the learner might construct generalities and abstract the underlying concept (English & Sharry, 1996). English and Sharry felt that analogical reasoning plays an important role in this process. Reasoning by analogy is defined as a mapping from a base to a target (Gentner, 1983). For example, in mathematics the analogy "4 is to 6 as 10 is to 13" the base is "4 is to 6" and the target is "10 is to 13." Elements in the base are mapped into elements in the target. 4 is mapped into 10 and 6 into 13. The relation of "less than" in the base then corresponds to the same relation in the target. When using concrete representation systems in mathematics the base is the concrete representation and the target is the concept to be learnt (English & Halford, 1995). Before making use of analogical transfer one must notice the correspondence between the target problem and base problem and retrieve the base as a generalised structure. The role of analogical reasoning in the algebraic domain is yet to be delineated.

When transferring between tables of data and visual patterns, students need to map the top row of the table to the step number in the pattern and the bottom row to the number of students, that is mapping the base (the table) to the target (the visual pattern). Many students experience difficulties with this process. This failure to notice the critical sameness between situations is particularly apparent when transfer of learning fails to occur. English and Halford (1995) provide some insights into why these difficulties might occur. Firstly, the students could be mapping the problem into an inappropriate scheme. This could depend on either the materials used to represents the concepts or on the difficulties they experience with reasoning analogically. Secondly, the processing load imposed by the proposed mapping may be such that some students find it difficult to reason by analogy.

The literature identifies other specific reasoning processes that appear to enhance associating representations. One of these is the ability to think flexibly (Warren & English, 1995). There seems to be two different interpretations of the concept of flexibility. The first refers to how students perceive geometric shapes and visual patterns. If their perception of the shape or pattern hinders their ability to visualise the shape or pattern in differing orientations or within more complex patterns then they are judged as inflexible (Tartre, 1990). The second refers to how students approach a problem solving situation, that is, their willingness or unwillingness to change their approach to the problem (Chancellor, 1991; Krutetskii, 1976). Lipman (1985) believed that some of the key processes crucial to flexibility are spatial thinking including a facility to mental rotation, logical and analogical reasoning, classifying and hypothesising, and an ability to complete patterns and generalise. The role these reasoning processes play in early algebraic experiences need to be examined.

The aim of this study was to begin to delineate why difficulties are occurring with these approaches and to attempt to identify specific student characteristics that seem to support these approaches.

Methodology

The design for this study consisted of two components, a correlational design followed by a clinical interview. Research that uses more than one method to collect relevant data is often referred to as a multiple measurement approach. Bewer and Hunter (1989, p. 17) suggested that the fundamental strategy of a multiple measurement approach is to "to attack a research problem with an arsenal of methods that have non-overlapping weaknesses in addition to their complementary strengths." One aim of the clinical interview was to probe the thinking students engaged in when reaching generalisations. It also served to explore important areas that are inherent in using the patterning and table of data approaches for introducing the concept of a variable, for example, do students identify the common structures inherent in these approaches.

Written test

The written test consisted of three components, namely, generalising from visual patterns, generalising from tables of data, and understanding the concept of a variable. Both the components for measuring the ability to generalise from visual patterns and tables of data consisted of four questions. Each question focused on ascertaining the students' ability to look at visual patterns or tables of data and make appropriate generalisations. Each represented the four possible linear generalisations that can be made from visual patterns, that is, x + c, ax, ax + c, and ax - c where 'a' and 'c' are constants and positive integers, and x is the variable. The functions were chosen such that each patterning question matched a corresponding table of data question. The questions chosen for the written test were as follows:

Patterning component

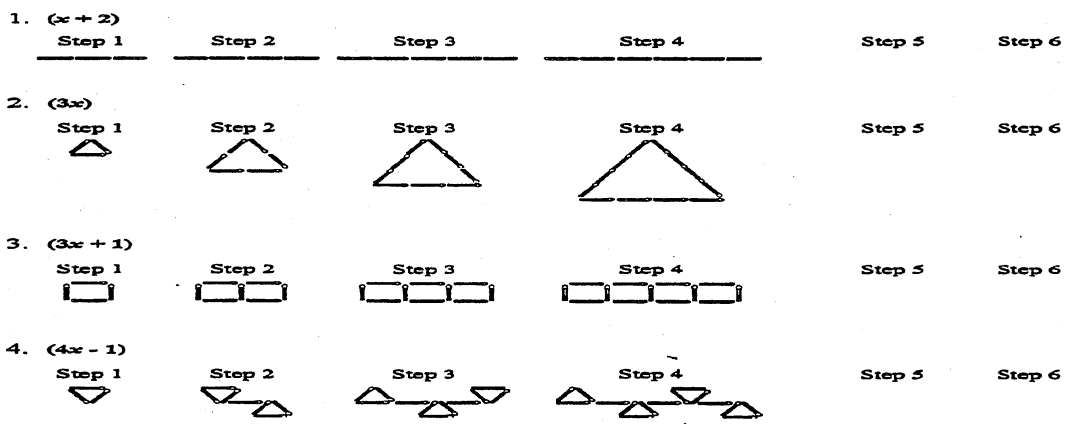

Table component

1. (x+3)

Input	1	2	3	4	5	6	7
Output	4	5	6	7	8		

2 (2x)

Input	1	2	3	4	5	6
Output	2	4	6	8	10	

3. (3x+2)

input	1	2	3	4	5	6	7
Output	5	8	11	14	17	20	

4. (2x-1)

Input	1	2	3	4	5	6
Output	1	3	5	7	9	11

Each question for each component began with the following set of instructions:

(1) Continue the pattern for 2 more steps
(2) How many sticks would you need to step 30? *Or* What number would the computer turn 30 into
(3) Write your rule in your own words
(4) Write your rule in symbols.

Semi-structured interview

The function chosen for patterning question was the same as question 3 from the written test, that is $3x + 1$. Students were asked to make a square using sticks. They were then asked to make two squares, three squares, and so on. The patterns students were required to generate were:

The patterning approach is used not only used to introduce the concept of a variable but also to explore the notion of equivalence. With this approach, students are required to physically construct and reconstruct the pattern in a variety or ways, each leading to new expressions of generalisation. For example, the above pattern can be described as both $3n + 1$ and $4 + 3(n - 1)$. Thus these expressions are equivalent as they represent the same pattern. In the interview, students were asked to: describe the pattern in their own words; represent their descriptions with concrete materials; ascertain how many sticks were needed for 30 squares; describe the general pattern using their own words; express the general pattern using algebraic symbols; reconstruct the pattern using another method; describe this second method in both words and symbols; and relate the two expressions generated by the two different methods. As one of the aims of the interview was to delineate students ability to transfer between the two representations, the linear function chosen for the table component was also $3x + 1$.

Sample

The sample comprised of 379 students, aged between 12 years and 2 months and 15 years and 10 months. Both schools chosen for the study consisted of students from lower-middle socio-economic status, with a variety of ethnic backgrounds represented. Both schools were large metropolitan schools comprising up to two thousand students. The classes included in the main study were randomly selected from the total student population. The results from the written test were used to select students to be interviewed. The results for each component were ranked and three students were selected at the 1st, 25th, 75th and 100th percentile for each of the written components. These particular percentiles were chosen to highlight the difference between 'high achieving' and 'low achieving' students. In all 36 students were interviewed, 12 from each component of the written algebra test. Each interview was videotaped.

Results

Written test

The percentage of student with correct responses to each part of each question was calculated. From this data it seemed that students on the whole experienced greater success in articulating the rule (part (3)) than in writing the rule in symbolic form (part (4)). In order to compare the results of the patterning component with the results of the table component, a McNemar (1969) test was utilised. For this study, students' responses were matched according to the question number and whether the response was verbal or symbolic, for example, their verbal responses to question 1 of the patterning component was matched with their verbal response to question 1 of the table component and so on. A fourfold frequency table was established for each pair of data and a McNemar test was utilised to test for statistically significant changes between the cells. Table 1 summarises the results for the verbal responses.

Table 1

Comparing the Verbal Responses for the Table and Patterning Questions (N=379)

Question	Chi-square (Verbal)
1	16.15*
2	106.25*
3	1.92
4	10.25*

* $p < .001$

For Questions 1, 2, and 4 , for the verbal responses, the number of students who correctly answered the table question and incorrectly answered the patterning question was statistically significantly higher than the number of students who incorrectly answered the table question and correctly answered the patterning question. This seems to indicate that students found it easier to verbalise the generalisations in the table questions than the generalisations in the visual patterns questions. The trends for the symbolic responses mirrored those for the verbal responses. Thus, on the whole, students experienced a greater degree of success with the table component as compared with the patterning component.

Semi-structured interview.

Students' responses to the patterning and table of data questions were transcribed and examined in order to identify trends in the responses. On examination of the students' responses it seems that, as expected, students at 1st, 25th, 75th, and 100th percentiles exhibited a growth in differing characteristics. These characteristics related to students' ability to: deal with number (arithmetic competence); initially see the pattern; articulate their thinking; identify the generalisation, with a particular emphasis on the strategies used; express the generalisation as an algebraic expression; and think flexibly, that is, their willingness to change their approach to a problem.

Low achieving "patterning" students, that is, students at the 1st and 25th percentile, tended to make arithmetic errors. For example, when ascertaining the number of matches needed for 4 squares, Alison said, *There are 4 for one square and you just keep adding 3 that makes 15, no 16.* They displayed difficulty in initially seeing the pattern and articulating their thinking. They tended to see the pattern in terms of the ratio concept or the additive strategy, that is, *If 10 sticks (were) needed for 3 (squares) so just times 10 for each 3 (squares)* (the ratio concept) or *Start with 4 for the first square and add 3, add 3, add 3, and so on* (the additive strategy). Low achieving "patterning" students were unable to express their generalisations algebraically, and exhibited an inability to change their approach to the problem, that is, to think flexibly. For example, Matthew wrote '10 in 3' when asked to express his expression in symbolic form, and once Kaylin identified the additive strategy she could not go beyond it and thus was incapable of finding the correct functional relationship. Low achieving patterning students also exhibited an inability to reconstruct the pattern by another means.

Low achieving "table" students shared many of the same characteristics as their patterning cohorts. They did show a greater capability for articulating their thinking as compared with their counterparts on the patterning question. For example, for the table question, Simon said, *the top line goes up 1 each time and the bottom line goes up three each time.* By contrast, for the patterning question Alison said, *Divide 4 by 30. Because each time the first square is 4 and you keep adding 4 on so I got to 28 and that gave 105* *and half it and so it's 2.* This supports the finding from the written test that students seemed to find it easier to verbalise the generalisation in the table questions.

High achieving "patterning" students, that is, students at the 100th percentile, exhibited a high level of arithmetic competence, recognised the generalisation at an early stage and articulated their thinking in a manner that aided expressing the generalisations in symbols. For example, Bede said, *Each time a stick is added to the top and the bottom and then one on the side and it's repeated. (To work out the number of sticks (sic)) Amount of squares by 3 add 1.* They exhibited a high degree of flexibility in their thinking (i.e., were willing to change their approach to the problem). For example, when Bede was asked if he could reconstruct the pattern by another method, he said, *Take all the centres out (leaving the 1st and last vertical) so you have a rectangle and then fill the sides in to make squares.* They tended to break the pattern into differing repetitive parts and reconstruct the pattern using these parts. Not only could they all recreate the pattern in a number of ways but also could express each method algebraically and recognise that all of expressions they generated were equivalent.

High achieving "table" students, that is, students at the 100th percentile, also exhibited a high level of arithmetic competence, and recognised the generalisation at an early stage. They all seemed to select a pair of data and attempted to find the relationship by trailing addition and multiplication. They exhibited a reluctance to trial subtraction. Once they found a relationship that successfully linked one pair they quickly checked it by considering another pair of data. If their initial 'guess' was incorrect they quickly found another relationship. This approach contrasted with the low achieving "table" students. Once these students identified an initial function they exhibited a reluctance to change either the coefficient of x or the constant. For the high achieving students, an inability to recognise the correct functional generalisation usually resulted in their not being able to attempt the question at all. They did not use either the additive strategy or the ratio strategy.

All students were asked to complete both the patterning question and the corresponding table question. None of the low achieving students in each group successfully completed either question. While the high achieving "patterning" students were all able to complete the table question, only one of the high achieving "table" students could identify the correct generalisation for the patterning question. She could also reform the pattern in differing ways, that is, flexibly manipulate the pattern. She said, *Times the sides by 2 (to obtain the number of horizontal sticks needed for the pattern) and add the number plus one (to obtain the number of vertical sticks needed for the pattern).* She then said, *You could multiply one less than the number of squares by 3 and add 4 for the start (the first square).* The other two high achieving table students failed to exhibit the same degree of flexibility.

In summary, it seemed that students, on the whole, experienced a greater degree of success on the table component than on the patterning component. The low achieving table students shared many common characteristics with their patterning cohorts. Students in both groups made arithmetic errors, had difficulty in expressing their generalisations in symbols, and were unable to think flexibly. But the low achieving students in the table component experienced a greater degree of success in articulating their thinking.

To be successful in the patterning component it seems that students required some added characteristics above and beyond those exhibited by their table cohorts. They needed to be able to continually manipulate, both physically and/or visually, the materials to form new generalisations. This flexibility in thinking not only allowed them to understand the notion of equivalence but also to see the common structures between the patterning question and the table question. They were the only group who could successfully map one problem onto the other, that is, reason analogically. A typical response was, *the input number in the table is the same as the number of sticks*, thus successfully mapping the relational components of each question. Their table cohorts were unable to do this. Another added attribute of the successful patterning group was that they continually related their generalisation to the problem under consideration. This was reflected in the permissible values they assigned to the variable in their generalisations. They continually checked if the values resulted in 'imaginary' situations, such as, a negative number of boxes.

Discussion of findings

Many of the findings in this study confirm and extend the results found in this recent research. As indicated by Stacey and McGregor (1995), language seems to play an important role in the generalising process. The present findings confirm the view that students not only find it easier to articulate their generalisations than to describe the generalisation symbolically (Kaput, 1992; Stacey & MacGregor, 1995; Yerushalmy & Shterenberg; 1994) but also find it easier to describe generalisations in tables of data than in visual patterns. The expression of patterns as numbers seemed to assist students in their search for numeric relationships. This could reflect a change in the processing load imposed by these tasks (English & Halford, 1995).

The literature does not appear to delineate characteristics of successful and unsuccessful students in this area of research. Orton and Orton (1994) acknowledged the importance of arithmetic competence and Stacey and MacGregor (1995) acknowledged the importance of correct verbal descriptions. The results indicated spatial visualisation, that is, the skill of mentally manipulating, rotating, twisting and inverting pictorially represented stimuli (Tartre, 1990), seemed important for generalising from visual patterns. For example, successful students manipulated the materials to form patterns and recognised the pattern being generated. They also broke the pattern into repetitive parts and reconstructed the visual pattern in a variety of ways. All of these processes seemed to involve spatial reasoning.

Analogicial reasoning also seems to play a significant role in successfully transferring between and linking representations. In the semi-structured interview, students who linked the two questions exhibited an ability to map the relational structure of one problem onto the other (i.e., reason analogically). The number of students who failed to link the patterning question with the table question confirms Lesh, Post and Behr's (1987) belief that students rarely seem to get things into a single coherent representation.

The results of the semi-structured interview intimated that both dimensions of flexible thinking were important for generalising from visual patterns and tables of data. Successful students seemed to be able to perceive the visual patterns from differing perspectives (Tartre, 1990). They also quickly changed their approach to reaching a generalisation once they realised that their generalisation was not applicable for all the cases presented. For example, in the semi-structured interview, high achieving students seemed to search for a generalisation by considering one pair of data at a time and then tested if the relationship fitted all other pairs. If the relationship was incorrect, high achieving students quickly refocussed on the original pair and discovered another correct relationship.

This research gives initial insights into why students are experiencing difficulties. Both approaches seem to require an array of reasoning processes not generally believed to be needed for early algebraic understanding. For example, the role of spatial reasoning in the algebraic domain is not generally acknowledged in the teaching of algebra. The intention behind these approaches is to introduce the concept of a variable in a more meaningful way. As this research begins to indicate, such approaches may be so complex that they hinder rather than help understanding.

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