THE EFFECTIVENESS OF A NO-GOAL APPROACH IN SOLVING PYTHAGOREAN 2-STAGE PROBLEMS

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Four groups of Year-8 students completed a set of 2-stage Pythagorean problems under differing conditions. One group completed the set with specific goals (find x) without instruction; the second and third groups were encouraged to either work forwards or backwards from the goal (x); whereas, the fourth group completed the set with unspecified goals (no x's). On a subsequent set of test problems, the no-goal group demonstrated a superior performance than the other groups in applying Pythagoras' Theorem.

Introduction

In the early stages of learning about the theorems and rules of geometry and trigonometry, many students follow a traditional sequence of instruction. In the introduction of formal geometry for example, students often begin their study by learning about the parallel line theorems (alternate, corresponding and co-interior). At first, the learners will apply these theorems to a number of elementary problems which only require the application of one theorem. Following this phase, students will be required to solve problems which need several applications of the theorems. Many of these types of problems can be classified as transformation problems (see Greeno, 1978), where an initial state is transformed to a specific goal state. In order to solve a transformation problem, a subgoal or a number of subgoals need to be calculated before the goal state can be found. For many learners, these elementary geometry problems may be their first experience of mathematical transformation problems.

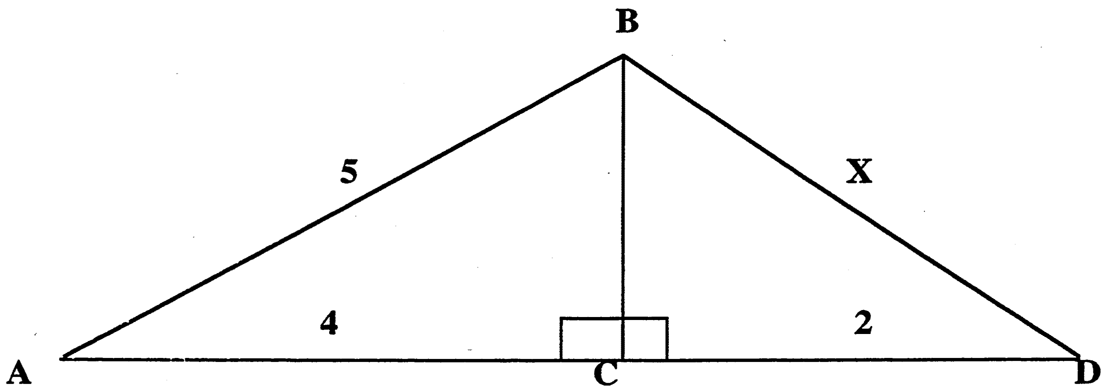
To solve transformation problems, novice problem solvers rely heavily on backward-working strategies such as means-ends analysis (see Mayer, 1983). Although means-ends analysis can be an effective problem solving strategy, its use creates a high cognitive load, as problem solvers constantly need to refer back to the goal or subgoals regardless of what position they occupy in the problem. Research conducted by Sweller and his collaborators (see Sweller, 1988, 1994; Ayres & Sweller, 1990) has shown that instructional phases which require problem solvers to use means-ends analysis are not particularly effective. Although problems may get solved, the strategy imposes a heavy cognitive load which may interfere with schema acquisition.

Much of the research conducted into the effects of using means-ends analysis and its relationship to learning was conducted in the 1980's, and the results of these studies were influential in the development of cognitive load theory (see Sweller, 1994). A main goal of cognitive load theory is to design instructional techniques which facilitate schema acquisition by creating learning environments which reduce the demands made on working memory. Use of no-goal or goal-free problems is one instructional design which has been investigated by researchers in a number of different domains such as dynamics (Sweller, Mawer & Ward, 1983) trigonometry (Owen & Sweller, 1995), geometry (Ayres, 1993) and biology (Vollmeyer, Burns & Holyoak, 1996). A goal-free environment is created by removing specific goals in a problem set and asking problem solvers to find all unknowns rather than specific goals. By removing specific goals, the use of means-ends analysis is minimised (if not eliminated), which in turn reduces cognitive load.

Typically in goal-free studies, groups of subjects who completed no-goal problems were found to perform significantly better than groups who completed traditional goal-specific problem. In a study by Ayres (1993) in the domain of geometry, the no-goal group made fewer errors than the specific-goal group. Furthermore, no-goal subjects were more able to navigate a path through the problem space. On these two-stage transformation problems, requiring the calculation of a subgoal and a goal, many subjects in the specific-goal group could not progress as far as finding the subgoal. Successful navigation through the problem space is a fundamental requirement for the development of problem solving skills in geometry and trigonometry. The Ayres (1993) study found that no-goal problems aided this development, whereas undirected problem solving strategies, employing heuristics such as means-ends analysis, did not. Earlier research into expert-novice differences found that experts in a particular field tend to work forward from the givens, whereas novices work backwards (see Larkin, McDermott, Simon & Simon, 1980) from the goal, using strategies such as means-ends analysis. Instructional phases which promote the use of backwards-working strategies appear to retard the development of expertise. However, it is worth considering whether learning would be enhanced if novices could be instructed with forward-working strategies.

Consider the problem shown in Figure 1. This is a two-stage transformation problem requiring two applications of Pythagoras' Theorem with the subgoal (side BC) to be calculated before the goal (X) can be found. To work forward, a problem solver would calculate the subgoal (BC) first without consideration of the goal (BD). The problem is very restricted, as there are no other unknown sides to calculate, and therefore the solution path is fairly simple to find. The rule "find the side which joins the two small triangles together first" will identify the subgoal for problems of this particular configuration. As a consequence of having a non-complex solution path, instructional techniques may be able to induce a forward working strategy more readily in this domain. It is also conceivable that backwards-working strategies may also be effective. A simplified solution-path with few feasible alternatives may reduce the level of cognitive load sufficiently to enable backwards-working strategies to be effective. Evidence to support this argument was found by Ayres & Sweller (1990) and Ayres (1993) in a two-stage geometry domain. Problem solvers were more likely to find the solution-path if the problem configurations involved were familiar or simplistic in nature.

Figure 1: Example of a 2-stage Pythagorean Problem



This study was designed to not only investigate the effectiveness of no-goal problems in the domain of Pythagoras' Theorem, but to also explore the effectiveness of forward and backward-directed strategies. Four groups of subjects completed different acquisition phases. One group of subjects was encouraged to work forward, another backwards; a third group completed a set of no-goal problems, while the last completed a traditional problem solving exercise with no instruction phase other than undirected problem solving.

Subjects

Fifty six Grade Eight girls of average mathematical ability from a Sydney High School participated in this study. They had previously been taught the Theorem of Pythagoras and had completed problems requiring single applications of the theorem only. The students had little experience in solving transformation problems other than in an elementary geometry domain.

Materials and Procedure

Two sets of twelve problems were designed in a similar fashion to the two-stage transformation example depicted in Figure 1. The first set of problems was used as an acquisition phase, while the second set was employed as a test set. Both sets were presented on paper. Each problem consisted of a diagram which contained two unknown sides. To calculate these unknown sides two applications of Pythagoras' Theorem were necessary. Furthermore, one side (the subgoal) had to be calculated before the second side (the goal) could be found. In each case the subgoal was common to both triangles. Using Pythagoras' Theorem to calculate an unknown side in a right-angled triangle leads to two possible algebraic outcomes. If the unknown side is the hypotenuse then an equation will be formed with an expression which contains a plus: $x^2 = a^2 + b^2$ for example; otherwise the unknown side leads to an expression which contains a minus; for example: $x^2 = a^2 - b^2$. Both problem sets were generated so that each of these two expression types occurred an equal number of times at both the subgoal and goal positions. A common error made by students working in this domain is to apply one of these expressions to the wrong situation; for example, adopting the formula which has a plus when the negative is required. By counterbalancing the problem-sets, results would not be biased by a student who routinely favoured one of the two formulae only.

The four instructional methods were designed as follows. To encourage a forward working strategy, a group of subjects was asked at the beginning of each problem in the acquisition phase to, "Calculate the first unknown side you can". It was thought that by drawing attention to the subgoal initially, rather than the goal, subjects may be encouraged to work forwards. After finding this side, subjects were then told, "Now calculate x". In contrast, to encourage a backward-working strategy, a group of subjects were given the following instructions on each problem, "To find x we must first calculate an unknown side. Calculate this side". By focusing on the "x" initially it was thought that students may be directed backwards. After finding this side, subjects were then told- "Now calculate x". For the conventional group, subjects were told, "Find x" in all problems", and therefore had a specific goal task. For the no-goal group subjects were given the directions, "In each problem calculate as many unknown sides as possible". No mention of a goal (x) was made.

The subjects were randomly assigned to one of the four groups: *the Conventional Group*, the *No-goal Group*, the *Forward Directed Group* or the *Backward Directed Group*. Each group contained fourteen subjects. It should be noted that in other no-goal experiments in elemetary geometry and trigonometry (see Ayres, 1993; Owen & Sweller, 1985) significant effects were found following short acquisition periods. Subjects were given the acquisition set of problems and allowed twenty minutes to complete as many of twelve problems as they could in the time. Following this phase, subjects were given the second set of problems which were presented in the conventional goal-specific format with the instructions, "Find x in all of the following problems". Subjects were given thirty minutes to complete as many of these problems as they could. No feedback about correct solutions was given during the experiment.

Results and Discussion

For the acquisition phase, the mean number of problems attempted by each group was calculated (see Table 1). A one-factor ANOVA revealed that there was no significant difference between groups on this measure; $F(3,52) = 0.27$, $p = 0.85$. To observe how accurately subjects applied Pythagoras' Theorem in this phase, success rates were calculated. Success rates were found for each subject by dividing the total number of sides correctly found by the total number of attempts at calculating sides (there are two sides which can be found in each problem). For example, if a subject calculated ten sides in five problems but made one error in applying Pythagoras' Theorem, then this subject would have a success rate of 0.90. It should be noted that errors which resulted from basic slips of arithmetic, rather than fundamental misunderstandings of Pythagoras Theorem, were not included in this analysis, as the focus of this experiment was to examine the students' understanding of the theorem. The most frequent error (61%) was to use an algebraic expression with a plus rather than a minus (see above). The reverse error of using a minus when a plus was required accounted for 20% of all errors, while the remaining 19% were varied and could not be easily classified.

Table 1: Number of problems attempted and success rates for each group in the Acquisition Phase

	Number Attempted		*Success Rates*	
	Mean	*SD*	*Mean*	*SD*
Conventional	5.14	2.71	0.51	0.31
No-goal	4.86	2.14	0.75	0.29
Forward-directed	4.86	1.88	0.54	0.34
Backward-directed	4.43	1.65	0.54	0.34

A one-factor ANOVA revealed that there was no significant difference between groups on success rates; $F(3,52) = 1.51$, $p = 0.22$.

For the Test phase (see Table 2), both the number of problems attempted by each subject and the success rates were recorded in an identical method to those in the Acquisition Phase (see Table 1). Most errors (76.5%) were made by using a plus in the expression derived from the theorem rather than a minus. The reverse error of using a minus when a plus was required accounted for 11.2% of all errors, whereas the remaining 12.3% could not be classified. A one-factor ANOVA revealed that there was no significant difference between groups on the number of problems attempted; $F(3,52) = 0.49$, $p = 0.69$. However, there was a significant difference between groups on the success rate measure; $F(3,52) = 3.03$, $p < 0.05$. Post-hoc Fisher PLSD tests revealed that the no-goal group had significantly higher success rates (at the 95% level) than both the conventional and forward-directed groups. Comparisons between other pairs failed to produce significant results. Clearly, the subjects who experienced a no-goal acquisition period were more able to identify separate triangles within the problem configurations and use Pythagoras' Theorem appropriately in the Test Phase.

Table 2: Number of problems attempted and success rates for each group in the test phase

	Number Attempted		Success Rates	
	Mean	SD	Mean	SD
Conventional	7.79	2.22	0.60	0.31
No-goal	6.64	2.62	0.85	0.20
Forward-directed	7.50	3.03	0.60	0.26
Backward-directed	7.43	2.50	0.63	0.28

To investigate the improvement in success rates between the instructional phase and the test phase , one-tailed t-tests (paired) were conducted between the two phases for each group. One-tailed tests were used as it was expected that students would show some improvement as a result of the acquisition phase. The no-goal group significantly improved from 75% in the acquisition phase to 85% during the test phase ($t(13) = 2.37$, $p < 0.05$). As the test phase consisted of conventional "x-format" problems, there is a strong indication that the acquisition phase was good preparation and allowed subjects to transfer easily to the test format. For the three other groups all showed an improvement in performance but not significant: the backward-directed group increased from 54% to 63% ($t(13) = 1.18$, $p= 0.13$); the conventional group increased from 51% to 60% ($t(13) = 1.54$, $p = 0.08$); and the forward-directed group increased from 54% to 60% ($t(13) = 0.81$, $p = 0.43$). Clearly, the forward-directed group did not improve significantly from the acquisition phase to the test phase, indicating that the forward-directing strategy was not very effective. In contrast, the p-values of the conventional and backward-directed groups may be approaching significance. However, even if real effects were present for these two groups, their effectiveness compared with the no-goal group was poor.

Conclusions

Consistent with previous research (Ayres, 1993; Owen and Sweller, 1985), a no-goal approach has been shown to be a very effective method for learning in domains which require solutions to simple transformation problems. Removal of the goal appears to simplify the problem-space searches and perhaps reduce cognitive load, which in turn, may enhance learning. Furthermore, Silver (1990) argues that goal-free problems may encourage conjecturing which is fundamental to learning. Attempts to induce either a forward or backward working strategy only succeeded in producing results at the same level of performance as a conventional approach. Although, these attempts lacked sophistication, they nevertheless provided a further contrast to the standard no-goal and goal designs of other studies. However, it should be noted, that in both cases, little data emerged which indicated whether the simple instructional formats achieved their objectives of stimulating forward or backwards-working strategies. A further study replicating this experiment could collect qualitative data to discover how students react to such instructions. Such a study might lead to more sophisticated ways to induce a forward-working strategy. In summary, even though the experimental design of this study has not found any evidence to support the argument that learning might be enhanced by instructing novices in forward-working strategies, more data has emerged to demonstrate the impact of a no-goal approach.

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