# Representing the connectedness of mathematical knowledge

**Mohan Chinnappan[1], Michael Lawson[2] and Rod Nason[1]**
**Queensland University of Technology[1] and Flinders University[2]**

An important area of mathematics teachers' expertise is their content knowledge. One way to characterise the quality if this knowledge is to examine the organisation of the various components that form the content knowledge base. In this paper we describe the procedures we have used to generate estimates of the state of connectedness of mathematical knowledge of teachers. We illustrate the procedures by focusing on a single teacher's knowledge of the concept of square.

## Introduction

During the past ten years the role of mathematics teachers has changed considerably. The current thrust in teaching strategies places emphasis on understanding mathematics by helping students make sense of what is taught in the class. These changes are articulated in the recommendations such as that there is a need to 'shift from dispensing information to facilitating learning' (National Council of Teachers of Mathematics, 1989, p. 128). This change in emphasis for teachers' actions recognises the crucial role that teaching plays in influencing what knowledge is constructed by students. Even though the ultimate control of the knowledge construction process rests with the student, it is clear that the actions of the teacher in a lesson do have substantial impact on the outcome of that process.

In the wider project in which this study is embedded we are interested in drawing links between the knowledge and actions of mathematics teachers and the mathematical knowledge and actions of their students. In pursuing this objective we have examined the knowledge states of teachers and students, observed the planning and delivery of lessons and then investigated the knowledge states and problem solving performance of students. We are interested to see how the actions of the teacher might facilitate the establishment of a well-connected structure of knowledge in students. We have assumed that the generation of these connecting, or linking, actions by the teacher is affected to a substantial extent by the state of connectedness of the teacher's own knowledge. In this paper we describe the procedures we have used to generate estimates of the state of connectedness of mathematical knowledge of teachers. We will illustrate the procedures by focussing on a single teacher's knowledge of the concept of square.

## Background to the study

In recent years a number of studies have begun to address the above question by focussing their efforts on providing a rich description of knowledge and skills that teachers bring to the classroom. A review of the discussions of the quality of teaching (Merseth, 1993) and of the research literature on expert and novice teachers of mathematics (Leinhardt, 1989) clearly indicates that teachers need to be able to access and exploit three types of knowledge both before and during teaching.

The first type of knowledge is knowledge about learning, about learners and about the context of learning. This includes information that the teachers hold about the processes of learning, knowledge of learners' repertoires of conceptual and procedural knowledge about the topic (Leinhardt et al., 1991; Shulman & Sykes, 1986; Tamir, 1988); knowledge of the learners developmental states and conceptions of the learners' motivational patterns (Leinhardt et al., 1991); and their conceptions about learning of the subject matter (Pramling, 1992).

The second type of knowledge required by the teacher is knowledge about how to teach the mathematical content to the learners. Pedagogical content knowledge (Shulman & Sykes, 1986) includes components such as : understanding of the central topics in each area of subject-matter as it is generally taught to children of a particular grade level; knowing the core concepts, processes and skills that a topic can convey to the students; knowing what aspects of a topic are most difficult for the students to learn; knowing what representations are most effective in communicating the appropriate understandings or attitudes of a topic to students of particular backgrounds; and knowing what student misconceptions are likely to get in the way of learning (Tamir, 1988); and the teacher's perceptions and beliefs about the role of a teacher and the teacher's perceptions about students. Finally the teacher must have a store of appropriate mathematical content knowledge. It is this mathematical content knowledge that is the focus in this paper.

Mathematical content knowledge

Included in the teachers' repertoires of subject-matter knowledge are: (1) substantive mathematical knowledge, such as facts, ideas, theorems, mathematical explanations, concepts, processes (and connections between these elements) (2) understanding of knowledge about the nature and discourse of mathematics, (3) knowledge about mathematics in culture and society, and (4) dispositions towards the subject (Ball and McDiarmid, 1990).

Both the quantitative and qualitative characteristics of this knowledge are of interest. In a crude sense, the breadth of mathematical content knowledge has relevance for any assessment, since the mathematical actions of the teachers will be influenced by what knowledge they have available. So quantity of knowledge is important.

However, sheer quantity of knowledge is not all that is of relevance in influencing teaching action. The state, or quality, of organisation of that knowledge also makes an impact on the accessing and use of knowledge. Poorly organised knowledge may well not be accessed at the appropriate time during teaching or problem solving (Prawat, 1989; Schoenfeld, 1985). Lawson and Chinnappan (1994) showed that a substantial part of problem-relevant knowledge could well remain inert during problem solving. In that research the problem solvers for whom access was difficult also showed evidence suggesting that the extent of connectedness of knowledge was limited compared to that shown by successful problem solvers.

Knowledge connectedness

The quality of connections among knowledge components is assumed to influence the ease with which the presence of one element aids in the retrieval and use of another in a problem environment. Schoenfeld (1985) observed that development of mathematical thinking requires not only mastering various facts and procedures, but also understanding connections among them. He suggested that there is value in providing detailed descriptions of these structures that support such thinking. As yet we have very few detailed discussions of the ways in which the quality of connectedness of elements of knowledge can be investigated and represented.

Mayer (1975) represented the accumulation of new information in Long Term Memory as adding new 'nodes' to memory and connecting the new nodes with components of the existing network. He utilised this nodes-network framework to examine learning outcomes along three dimensions. In two of these dimensions, the notion of connectedness was employed to evaluate prerequisite knowledge and the activation of assimilative knowledge structures. 'Internal connectedness' was referred to the degree to which new nodes of information were connected with one another to form a single well-defined structure. The degree to which new nodes of information were connected with information already existing in the learner's cognitive structure, was called 'external connectedness.'

In a further analysis of mathematical knowledge structure, Greeno (1978) noted that within the schema the learner could also be expected to establish connections of how the concept relates to specific problems and situations. This notion of connectedness can be seen as an attempt to establish a link between knowledge of a concept and knowledge of how to use that concept, thus being concerned with the proceduralization of declarative knowledge.

We have used the notions of internal and external connectedness as a conceptual framework for representing the qualitative characteristics of the teachers' knowledge. We see this as useful for two main reasons. First, it seems conceptually sound in that it allows us to represent the complexity of a knowledge base in a way that focuses on the state of organisation of that knowledge. Our second reason for use of this connectedness framework is that it provides a basis for establishing relationships between knowledge states and teaching actions. In relation to this second purpose we suggest that one of the key motivations of the teachers, and one of the most important sets of their actions, can be seen as attempting to facilitate the establishment of both internal and external connections in the knowledge bases of the students. In this way we argue that it makes sense to seek out relationships between the teachers' knowledge, those of their actions designed to improve students' knowledge connectedness, and the resultant state of connectedness of the students' knowledge base. Thus we will focus here on trying to represent the connectedness of a teacher's knowledge, using the square as a focus concept.

In representing this connectedness we will lean heavily on use of the concept map. Since their popularisation by Novak (eg. Novak & Gowin, 1984) concept maps have been seen as useful ways to represent the spatial structure of knowledge relationships (Lawson, 1994). Other related mapping systems have been developed for essentially this purpose, such as Chi, Hutchinson and Robin's (1989) network model of knowledge coherence and McKeown and Beck's (1990) semantic net representation of students historical knowledge. In this paper we will describe how we have used the concept map and its characteristics to generate an assessment of the organisational state of a teachers' knowledge of the concept of a square.

## Method

Participants

In order to get a pool of participants for this project we approached a number of organisations within the states of Queensland and Western Australia including the Education Departments, the Mathematical Associations. A male high school teacher who has twenty years experience in mathematics was among the group of volunteers involved in the project. The teacher was the head of the mathematics department of his school.

Tasks and procedure

Three tasks were developed for the purposes of assessing teacher's knowledge about geometry. The first task, Free Recall (FRT), required the teacher to talk aloud about any idea that he could associate with the topic of plane geometry such as shapes and their properties. The second task, the Problem Solving task (PST), consisted of four problems. All the four problems were related either to plane and/or coordinate geometry. The first problem involved the use of Pythagoras's theorem in a rectangular coordinate system. Solution of the second and third problems required an understanding of the relationship between properties of squares and rectangles, segmentation of their areas and gradient of straight lines in a rectangular coordinate system. The fourth problem, again necessitated an integration of knowledge of properties of square and right-angled triangle. The final task, Geometry Probing (GPT), was designed to obtain further information about teachers talked about during the Free Recall task. In the GPT the teacher was required to respond to a number of questions each of which aimed at probing what was said during the first session. The probes were designed to provide the teacher with opportunities to display his available knowledge of features of the target concept and his knowledge of relations between this target concept and a list of other concepts relevant to this area of mathematics. The checklists of features and related concepts that had been established in pilot work with expert mathematicians and are listed in Tables 1 and 2. For example, if the teacher mentioned symmetry of squares in the FRT but did not elaborate on this idea he was asked to explain this idea in the probe session. This strategy was expected to provide more data about what the teacher knew about the squares. The use of the above techniques have been argued to provide rich data about interconnections that exists between knowledge units, and their structure (Royer, Cisero & Carlo, 1993).

The teacher met with us on three occasions and undertook one of the tasks in each session, in the order indicated above. In the first session he was invited show his knowledge of geometry by talking, drawing and writing in response to the FRT. During the second session he was asked to complete the PST which involved generating solutions to four problems. He was encouraged to talk aloud during the solution attempts. In the third session we sought the teachers' responses to the probe questions. All three sessions were video-taped and transcribed for subsequent analysis.

The data analysis focussed on accumulating as much data as possible about the teacher's knowledge of squares. The use of three different contexts in which the teacher was required to search for knowledge helped maximise data generation. Two experienced teachers of mathematics analysed the transcripts and video-recordings. Any differences in their interpretation of the data was resolved in consultation with a third person who was also an experienced mathematics teacher and teacher educator. In order to generate a more complete picture about the extent of the teacher's knowledge we developed a concept map and devised a system for scoring of the concept map.

Structure of the concept map

The concept map was designed to allow identification of four types of information in the teacher's performance:

1. The features of the target concept, both the essential, or defining and other related features;
2. Relationships that were established within the features of a concept;
3. Relationships established between the target concept and related concepts.
4. Other representations of the concept, such as in analogies, illustrations or real life examples.

The concept map shown in Figure 1 was structured to record these types of information. Information related to the features of the concept was recorded in the bottom left corner and elaborations on these features were recorded in the lower right section of the map. Links to related concepts were recorded on the upper right part of the map. Type 4 information was included in the top left part of the map. Where relationships were expressed they were labelled on the map.

Scoring of levels of connectedness

Table 1 shows the types of judgements made by the teacher that were internal to the schema for square. These internal connections made about of the square were separated into two categories, *essential* and *related Features*. Essential features refer to properties that were defining properties of the target concept. Related features refer to properties that were derivatives of squares, such as area and symmetry of square.

The teacher's verbal and physical actions in the three tasks were examined for identification of the features and any elaborations, or expansions, on any of the identified features. In Table 1, the teacher mentioned that squares had line symmetry, which was a counted once under the column, Initial Link. The teacher proceeded to expand on this point about line symmetry by giving us four distinct elaborations about symmetry. This is noted in the Elaboration column.

In addition to counting the nodes, we were interested in making judgements about the quality of these connections. This was achieved by scoring the teacher's responses in relation to a scale of connectedness that had three levels. Level 1 connectedness recognised that features had been noted. A level 2 rating was given if a feature was both noted and elaborated upon. If the noted and elaborated features were further discussed by establishing properties of an elaboration a level 3 rating was recorded. The levels of connectedness are intended to represent the richness of connections associated with the target concept.

Any connections made between a square and other geometric figures were the subject of a separate external connection analysis. A significant aspect of a teacher's knowledge base could be expected to be concerned with relations between squares and other geometric figures such as triangles and parallelograms. The node frequency and levels of connectedness of these external connections were scored in the same way as internal connections.

## Results and Discussion

Concept map

Figure 1 shows the concept map that was constructed for the focus concept of square for our expert teacher, Gary. The most visible aspect of this map is that Gary has built up a reasonably large number of connections involving squares as shown by the number of arrows emanating to and from the focus concept. The bottom left hand corner of Figure 1 shows that his knowledge about the properties of squares is extensive. The top-half of the map indicates that Gary has been able to link squares with other polygons such as rhombus and isosceles triangles. Gary's understanding of defining properties of a square was extended to more advanced features such as symmetry and area. This component of his knowledge is depicted by the right-half of Figure 1. The upper left section of the concept map provides information relevant to the contexts in which the above knowledge components were activated and the ways that these would be expressed by this teacher.

The concept map of the expert teacher revealed that he has a rich store of knowledge as manifested by the multiple links he had constructed with many facets of a square. These links suggests that the teacher not only has a well-developed understanding about the properties of square but more importantly, his has an extensive network of connections between squares and other geometrical figures. For example, he was able to represent a square as a polygon and as a quadrilateral. A square was seen as a subset of rhombus, though the relationship was not seen as symmetrical. His ability to recognise a right-angled triangle as forming part of the square also indicated that the teacher was able to use basic properties such as right-angles to create other figures. Gary's knowledge base indicates that he was able to highlight differences and similarities that exist between square and related figures. That is, he has an advanced relational understanding of geometric figures.

Gary commented that a square has symmetrical properties and that this property can be created by more than one way. This understanding of symmetry is extended to his observations that squares tessellate. Here one can detect not only an understanding of the geometric properties but also practical applications of squares.

<u>Analysis of connectedness</u>

Results of analysis of the focus concept in terms of number of links and quality of the links are provided in Table 1 and 2. As explained earlier, Table 1 shows internal connections whereas Table 2 shows external connections made by the teacher. Gary made a total of eight links with information that was directly associated with square and five links with information which can be seen as extensions of his knowledge about square. Gary was able to access both knowledge related to each of the essential features of the square and knowledge of each of the related features.

The 'Level of Connectedness' column in both tables provides a measure of the richness of connections associated with this schema knowledge. This analysis suggests that Gary's knowledge of square is quite well elaborated. Five of the seven judgements about level of connectedness were at level 2, with one each being at levels 1 and 3.

The analysis represented in Table 2 suggests that Gary has established a wide range of links from square to related concepts. The only omission from his discussion was the description of square as a plane figure. The level of complexity of these external connections from square is generally lower than those observed for the internal organisation of knowledge about a square, with only two of the six judgements receiving level 2 or 3 ratings.

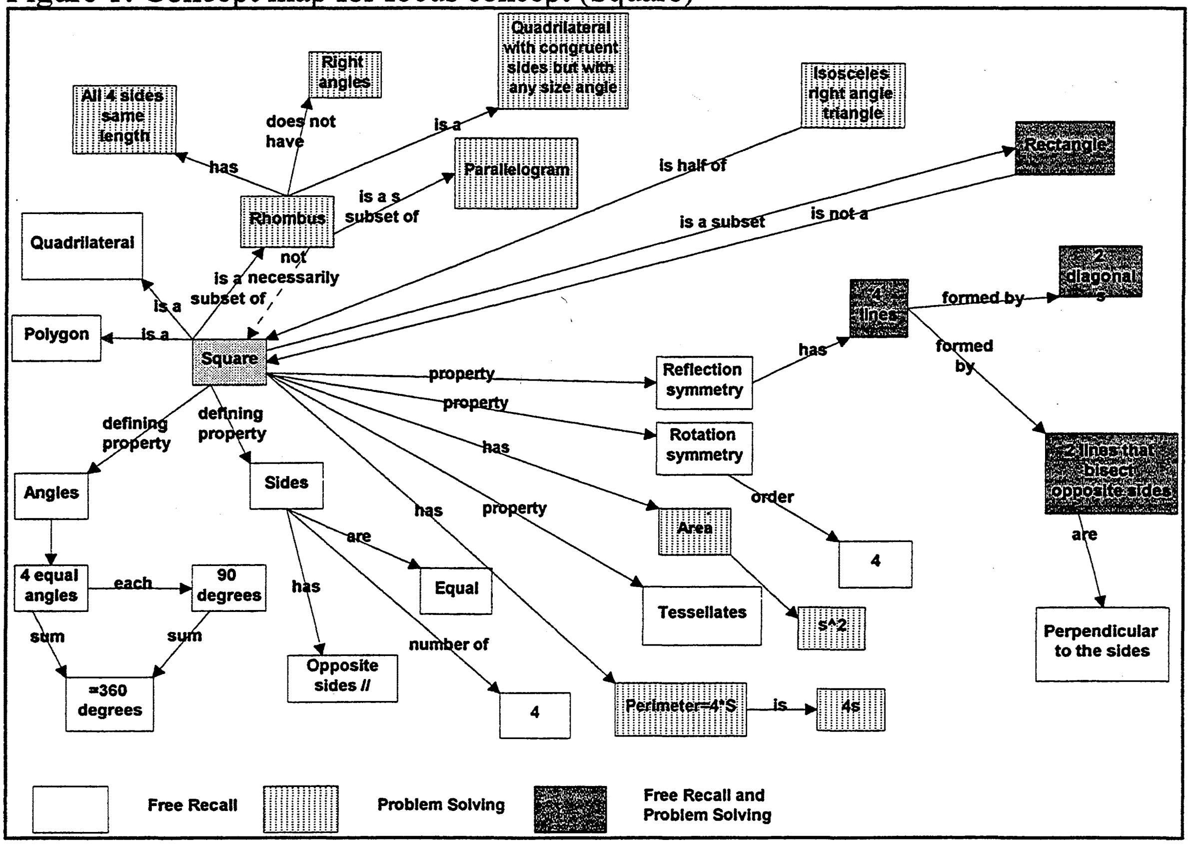
Table 1- Focus Concept: Square (Internal Connection)

| Features | Node Frequency | | Level of Connectedness |
|---|---|---|---|
| | Initial link | Elaboration | |
| Essential Features | | | |
| Angles | | | 2 |
| Four | 1 | | |
| All equal | 1 | | |
| All 90 degrees | 1 | | |
| | | | |
| Sides | 1 | | 2 |
| Four | 1 | | |
| Equal length | 1 | | |
| Opp. parallel | 1 | | |
| | | | |
| Sub total (essential) | 8 | | |
| Related features | | | |
| Area | 1 | 1 | 2 |
| Perimeter | 1 | 1 | 2 |
| Line symmetry | 1 | 4 | 3 |
| Rotational symmetry | 1 | 1 | 2 |
| Tesselates | 1 | | 1 |
| | | | |
| Subtotal (related) | 5 | | |
| TOTAL | 13 | 6 | |

Table 2 - Focus Concept: Square (External Connection)

| | Node frequency | | Level of connectedness |
|---|---|---|---|
| | Initial link | Elaboration | |
| | | | |
| Plane figure | | | |
| Rhombus | 1 | 4 | 3 |
| Right-angle triangle | 1 | | 1 |
| Rectangle | 1 | | 2 |
| Polygon | 1 | | 1 |
| Quadrilateral | 1 | | 1 |
| Parrallelogram | 1 | | 1 |
| | | | |
| TOTAL | 6 | 4 | |

Figure 1: Concept map for focus concept (Square)




## Conclusion

The purpose of the study reported here was to provide data about an expert teacher's knowledge about concepts in plane geometry. In the present study we focussed on knowledge the teacher has constructed about squares by providing three different contexts in which the teacher could have displayed his understanding of square and its applications. The focus of our representation of this knowledge was on recognition of both the presence/absence of details of the knowledge and judgment of the richness of connectedness of this knowledge , both internally and externally.

Experience with this procedure suggests that there is value in use of the idea of connectedness to make judgements about the qualitative features of a knowledge network. The procedures we have developed here extend the utility of Mayer's (1975) distinction between internal and external connectedness. The levels of connectedness framework allowed us to make a distinction between the richness of the internal and external connections associated with square that were made by this teacher. In a practical sense this analysis seems to have provided a way for use to establish a qualitative estimate relevant to the teacher's knowledge base that can be used in other parts of the project where relationships between mathematical content knowledge, pedagogical knowledge and teaching actions are of concern.

Our expert teacher has a wide body of knowledge related to squares and other geometrical figures in the area of the plane geometry. Beyond this quantitative judgement our analysis provides a basis for making statements about the quality of organisation of this knowledge. In this regard we suggest that this teacher's knowledge is richly organised, having a high degree of internal connectedness and a moderate degree of external connectedness. Analysis of the teacher's output in other parts of the interview, where other concepts were the target concepts, will allow us to gain more information that might be used to validate our judgement about the richness of the external connections. Experts organise their domain knowledge in meaningful clusters such as schemas (Marshall, 1995). Because experts construct multiple connections with core concepts, they can be expected to access and use that knowledge in different contexts. Thus, organisation improves assimilation of new information and accessing of prior knowledge as evidenced by our expert teacher.

The variety of contexts in which the expert teacher could display a particular concept can also explain the richness of explanations such teachers are able to generate during the course of a lesson. Recent work in the area of arithmetic (Leinhardt, 1988) has shown that expert teachers draw on a well-developed content knowledge base in order to make inferences and generate explanations during the course of their instruction. The results of this study is consistent with the these studies.

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