Visual reasoning and teaching styles in mathematics classrooms

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Four mathematics teachers with different preferences for the use of visual strategies in solving problems are compared in their use and support of visual methods in their classrooms. Presmeg (1985) found that no teacher who displayed a strong preference for non-visual strategies in problem solving made substantial use of visual methods in their classroom. In this study, the teacher who least preferred to use visual strategies to solve problems displayed a strong use of visual methods in the classroom.

Introduction

Visual reasoning involves understanding a problem or concept in terms of a diagram or a visual image. Zimmerman and Cunningham (1991) provide a workable definition that emphasises both the physical and mental aspects of the visualisation process: 'Mathematical visualisation is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding' (p.3).

The use of visual images to assist in the understanding of mathematical concepts has become an area of renewed interest in mathematics education research. Some of this interest may stem from an acknowledgement that much pedagogical use has always been made of diagrams in mathematical discussions. There have been moves in recent years for an acceptance of visual methods beyond that of a pedagogical tool with Dreyfus (1994) arguing that 'the status of visualisation in mathematics education can and should be upgraded from that of a helpful learning aid to that of a fully recognised tool for mathematical reasoning and proof' (p.107). More recently, this promotion of the use of visual forms of mathematical reasoning has coincided with developments in the use of computer graphics and has led some to challenge the conventional presentation of mathematics. Tall (1991) notes that the concept of a limit appears to be 'an unsatisfactory cognitive starting point for the study of calculus' (p.110) and promotes the use of a graph-plotting program that can magnify at will small sections of graphs to develop the idea of the 'local straightness' of a curve.

Yet traps exist for the unwary in the use of visual methods. Presmeg (1986) notes three problems: '(1) The one-case concreteness of an image or diagram may tie thought to irrelevant details, or may introduce false data; . . .(2) An image of a standard figure may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram; . . .(3) An uncontrollable image may persist, thereby preventing the opening up of more fruitful avenues of thought ' (p.44). While some of the reluctance to accept visual methods may stem from the traps inherent in their use, Eisenberg and Dreyfus (1991) note a reluctance on the part of students to reason visually in calculus courses 'even if they are explicitly and forcefully pushed towards visual processing' (p.29). In a separate article, Dreyfus (1994) postulates that student reluctance may be the result of teachers' attitudes towards visual arguments and their use of them solely as introductory, accessory or auxiliary arguments. In other work, Presmeg (1985 and 1986) found a weak statistical relationship between a teacher's preferred personal use of visual methods to solve problems and the extent to which the teacher uses visual presentations when teaching mathematics. Presmeg (1985) defined 'the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and non-visual methods' (p.42) as the *mathematical visuality* of the person. She defines a mathematics teacher's *teaching visuality* as 'the extent to which that teacher uses visual presentations when teaching mathematics' (p.42). While Presmeg came to the overall conclusion that 'irrespective of whether or not they prefer to solve mathematical problems by visual methods, teachers may or may not use visual presentations in the classroom' (1985, p.269), she also observed that the removal of one teacher from a problems by visual methods, teachers may or may not use visual presentations in the classroom' (1985, p.269), she also observed that the removal of one teacher from a sample of thirteen teachers led to a significant statistical relationship being detected between mathematical visuality and teaching visuality (1985, p.144-145). In addition, 'no teacher of low mathematical visuality was found to be a visual teacher' (1985, p.269).

These reports of the possible connection between teacher attitudes towards visual methods and the support of such methods in the classroom have been the starting point for the present piece of research. This paper reports on a study of Canterbury secondary mathematics teachers and uses the work of Presmeg as a basis for analysing the teachers' personal use of visual methods in problem solving and the use of visual methods in the classroom.

Methods

The research proceeded in two stages. The first stage involved quantifying the mathematical visuality of teachers and using this to select four teachers of differing mathematical visuality for further case study. The second stage involved observing these selected teachers, interviewing them and drawing conclusions about their teaching visuality.

Sample

For the first stage a 'Problem Solving Questionnaire' was posted to 35 secondary schools in the greater Christchurch region. Responses were received from 22 teachers in 17 schools. For the second stage of the research, four teachers were identified as case studies using the results of the questionnaire. Two of the teachers displayed high mathematical visuality while the other two displayed low mathematical visuality.

Measuring mathematical visuality

The mathematical visuality of teachers was gauged by the use of a 'Problem Solving Questionnaire'. This consisted of eighteen problems that could each be solved either by visual or non-visual methods. Aside from a couple of minor variations in wording, the problems were the same as those used by Presmeg (1985) in her analysis of the mathematical visuality of teachers.

The responses to the Problem Solving Questionnaire were analysed and each question was assessed on the extent of the visual reasoning that had been used in the solution, ranging as scores of 2, 1 or 0. A score of 2 was given to a solution that used a visual method as an essential part of the solution. A score of 0 was given to a solution that used no visual method. Scores of 1 were rare and were given for a few solutions where there were some ambiguities in the methods used. These scores were totalled for each teacher to give a measure of the mathematical visuality of the teacher.

Measuring teaching visuality

The teaching visuality of the teachers was developed from the work of Presmeg (1985). In her thesis, Presmeg (1985, p.126-133) measured teaching visuality according to the relative occurrence or absence of the following classroom aspects: a classroom atmosphere that is controlled but relaxed and unhurried; a pictorial presentation of a topic; use of spatial arrangements in algebra; use of gesture (hand waving); use of instructions to form mental images; use of mobile models or images; instruction to pupils to use arm, finger or other body movements; use of concrete materials; use of colour; use and encouragement of intuition; use of pattern-seeking methods; delaying the use of symbolism; creating deliberate cognitive conflict; and accepting and demonstrating alternative methods. Further to this, Presmeg (1985, p.193-194) went on to observe teachers in some detail and itemised a further 45 behaviours in the classroom, most of which were noted in the visual teachers and absent in the non-visual teachers. So as to make the task of observation manageable, the most significant behaviours that were characteristic of visual teachers were included in the observation schedule for this case study: feels rushed because of a lack of time; uses humour; makes use of pupil efforts; uses 'real world' examples; stresses general methodological principles; shows awareness of other methods of solution; expresses feelings towards mathematics; uses language evocative of imagery; and values a visual presentation.

An observation schedule was constructed and trailed with two teachers who had not been selected as case studies for the second stage of the research. A final format was arrived at in which each behaviour was ticked if it occurred and room was left to add a comment to this to assist in the writing up of the observation.

Procedure

Copies of the 'Problem Solving Questionnaire' were posted to 35 secondary schools in the greater Christchurch region. Teachers were told that the questionnaire was part of a research project into problem solving strategies. They were not informed that the purpose of the research was to investigate the use of visual methods. In this way the responses would not be biased towards the display of visual methods in the solution of the problems. Teachers were also asked at this stage to indicate their willingness or otherwise to participate further by being observed in their classroom.

The responses were analysed and each question was assessed for the extent of the visual reasoning that had been used in the solution. Of the teachers that were available for the next stage of the research, two with the highest visual reasoning scores and two with the lowest scores were selected as case studies to be observed in their classrooms. The teachers with high visual scores will be referred to as Teacher H_1 and Teacher H_2 respectively and the teachers with low visual scores will correspondingly be referred to as Teacher L_1 and Teacher L_2.

The case study teachers were then observed teaching in their classrooms. Each teacher was observed for two lessons with year 10 students (15 years old), approximately three weeks apart. One teacher, Teacher H_2, was observed additionally with a year 9 class. Each lesson lasted 50 to 60 minutes. In the first lesson, each teacher was observed in a normal classroom setting teaching a class and topic of their choosing. For the second lesson, the teachers were asked to incorporate two of the problems from the questionnaire that both Teacher H_1 and Teacher H_2 had solved by visual methods, while Teacher L_1 and Teacher L_2 had solved by non-visual methods. It was hoped that the inclusion of this common material would also reduce some of the variables from the first set of observations.

A week after the second observation each teacher was interviewed. They were presented with their original solutions to the problems in the questionnaire and asked to explain their particular solutions to the two questions used in the second lesson. This led to other questions also being discussed. The teachers were also asked about how they went about constructing the approach they took to the second lesson that involved the use of these two problems. A conversation of about 30 minutes duration evolved, at the end of which the full purpose of the research was revealed to the teachers and summaries of the lesson observations were given to them.

Analysis of Responses to the Problem Solving Questionnaire

The responses to the questionnaire were analysed as detailed before with scores of 2, 1 or 0 used to indicate the extent of visual reasoning used in each solution. Out of a total possible score of 36 from the eighteen questions, the scores ranged from a low of 6 to a high of 17. The median was 11, the mean 10.8 and the standard deviation 2.3. Presmeg (1985, p.122) reported for her sample of 30 teachers a range from 3 to 26 with a median of 12. From this initial survey sample, six teachers indicated they were not available for any further aspects of the research. These included two teachers with the lowest visual scores (6 and 8) and one teacher with a high visual score (14). From the teachers that remained, two teachers with the highest visual scores (17 for Teacher H_1 and 13 for Teacher H_2) and two with the lowest visual scores (8 for Teacher L_1 and 9 for Teacher L_2) were then selected as case studies to be observed in their classrooms.

The four case study teachers were all female. This came about through accident, not design. Of the 22 responses received to the questionnaire, 13 were from male teachers and 9 from female teachers. The male with the highest visual score and the male with the lowest visual score both were unavailable for further aspects of the research.

There were only two questions which Teacher H_1 and Teacher H_2 solved by a visual method while, at the same time, Teacher L_1 and Teacher L_2 solved with no evidence of a visual method. These were questions 10 and 13:

10. If you place a round of cheese on a pan of a scale and three-quarters of a round of cheese and a $\frac{3}{4}$ kg weight on the other, the pans balance. How much does a cheese weigh?

13. The distance that a tourist travelled by train is 150 kilometres longer than the course he travelled by steamer, and 750 kilometres more than his journey on foot. Determine the length of his entire trip if it is known that the distance he covered on foot was one third of the distance he covered by steamer.

These two questions discriminated most clearly between the high and low mathematical visuality of the teachers. For this reason, these two questions were selected as the questions to be used by all four teachers with their classes during the second lesson observation.

The Classroom Observations

A qualitative approach was used in the analysis of the observational data from the classroom lessons. During each observation, case study notes were taken according to the headings available in the observation schedule. These notes were then used to write a description of each lesson. The lessons were then viewed together and themes that recurred were extracted under suitable headings. Only those headings for which behaviours were observed have been used in this summary. In addition, Presmeg combined five of these categories under the heading *Teaching without rules* : use and encouragement of intuition; use of pattern-seeking methods; delaying the use of symbolism; creating deliberate cognitive conflict; and accepting and demonstrating alternative methods. The summaries below use this heading to simplify the comparison of the lessons.

Topics of Lessons

In the first lesson, each teacher chose their own topic for the content of the lesson. Teacher L_1 shaped a lesson around line symmetry, Teacher L_2 used enlargement, Teacher H_1 worked in applications of relations and their graphs, and Teacher H_2 used statistics with the year 10 class and solutions of linear equations with the year 9 class.

For the second lesson the teachers were asked to incorporate questions 10 and 13 from the questionnaire in any way they liked. All the teachers chose to devote the whole of the second lesson to problem solving.

General Classroom Atmosphere

The classroom atmosphere with Teacher L_1,Teacher L_2 and Teacher H_1 was controlled but relaxed throughout both lessons. Each lesson progressed at a pace that was comfortable for the students - not at all hurried. There was a friendly liveliness between both Teacher L_1 and Teacher L_2 and their classes. The rapport between Teacher H_1 and her pupils is best described as a gentle warmth. The teacher expressed some affection for the class in her manner and her language and the class appeared to have reciprocal feelings towards the teacher.

During the first lesson, Teacher H_2 adopted a measured, controlled delivery with the year 9 class. This appeared necessary to keep one or two bouncy individuals in the classroom focused on the task in hand. In the second lesson the arranging of the students into groups became an opportunity for much off-task behaviour. The atmosphere in the room appeared to have deteriorated since the previous observation.

With her year 10 class, Teacher H_2 adopted a low-key but firm approach in the first lesson. The class had divided itself into several factions which had placed themselves within the room as far away from each other as possible. In the second lesson the atmosphere was more friendly. An invitation to reorganise and work in different groups caused no disruption. Indeed the class remained on task throughout the period and tackled the problems with some commitment.

Use of diagrams

Only one instance was noted during the first lessons of the drawing of an inessential diagram by the teacher. This occurred when Teacher H_2 started the solution of equations with the year 9 class. A suggestive diagram was introduced when the word 'equation' was written on the board and then underlined with a stylised triangular balance. Students were then asked to suggest why this had been done.

For the second set of lessons diagrams were much in evidence in all the classrooms. Their use was inessential as the problems could be solved without the use of diagrams. Yet the teachers were all encouraging of their use. It is particularly worth noting that when Teacher L_2 provided her class with an answer to question 13, it began with a diagrammatic representation of the information in the problem to assist in the solution of the problem. This was *not* the method she had employed in her response to the questionnaire.

Use of Gesture

In the first lesson with Teacher L_1, students were required to make shapes which had at least one line of symmetry. One student during the exercise asked if that meant 'you can fold it over in half, can't you?'. This visual interpretation was confirmed by the teacher who then gestured with her arms sweeping out an arc until her hands met in the middle. Talk by the students of 'reflection' and 'symmetry' in two separate incidents evoked from the teacher a similar hand waving response. In the second lesson she frequently used gestures while explaining aspects of the questions to individual students. On one occasion she emphatically cupped her hands firstly to the right and then to the left to help illustrate the content of the two pans on the balances. Another question introduced into the lesson involved the use of an 'old fashioned toaster' in which two slices of bread have one side toasted first. They are then removed, turned over and the other side is toasted. While explaining the action of the 'old fashioned toaster' she seemed to let her hands almost represent pieces of toast as her held them face down to indicate that one side only was toasted first and then she flipped her hands in a half-circle to illustrate the toast then being turned over.

Teacher H_1 used gesture in several forms. It was a mixture of the inessential 'dramatic' move - as in opening the arms slightly and hunching the shoulders when asking of a student 'how far below the surface?' - and some spatially significant gestures - as when pointing down when saying 'down'. Her hand would sometimes trace in the air when describing a linear relationship. Individual encounters with pupils at their desks were often accompanied with gestures. In the second lesson, while explaining the use of balances to some students, the teacher would interlock her hands and sway her elbows up and down to indicate the up-and-down motion of a balancing arm with the pivot being at the interlocking hands. While talking of a weight being put on to a balance, the teacher would motion her hand downwards as if she were dropping something.

Neither Teacher L_2 not Teacher H_2 made significant use of gesture in their lessons.

Use of Concrete Materials

Only Teacher L_1 required students to use concrete materials in the first lesson. The main activity of the lesson required the students to cut out and use three shapes - a square, two squares joined, and four squares joined in the shape of a T - to make as many shapes as possible which had at least one line of symmetry. She also encouraged the use of concrete materials to help solve the problems in the second lesson by providing counters and using such statements as 'Yes, the counters are there, you can use them', or 'Just come and grab a handful - how ever many you think you will use.' Students displayed some dexterity in using and manufacturing materials to assist in the solving of problems.

The only other incident involving the use of concrete materials occurred, almost by accident, in the second lesson with Teacher H_1. She distributed giant jellybeans, four per group, as an encouragement to the students to engage in the tasks. Some groups used the jellybeans as concrete materials to help them calculate the answer to question 10.

Teaching without rules

All the teachers encouraged students to use their intuition. This was particularly evident during the second lesson for each teacher where students provided the answers for all the problems with a minimum of assistance from the teachers. As an example, when one student asked Teacher L_1 'How many counters do we need?' to solve one of the problems, she replied 'That's up to you to work it out'. When another student wanted to know if they had the right answer, the teacher refrained from giving it and reflected the problem back to the student with the question 'Does it work? Does it fit?' The general methodological principles of problem solving were frequently stressed, with lists of strategies being written on the board as students used them. Alternative methods were encouraged by all teachers with such phrases as 'Did anyone do it any other way?' or 'Is there more than one answer?' In this respect the efforts of Teacher L_2 are worth a comment. For her 'warm up' question, three solutions provided by the students were written on the board. All were done by trial and error by the students. It is worth noting that Teacher L_2 stayed with the students' attempts to solve the problem by this method for some time, thereby delaying the use of symbolism before, almost as a last resort, prompting the students with the remark: 'We've been trying trial and error. It hasn't totally worked, has it? Can we use algebra?'

Similar instances occurred for all teachers during the first lesson. Teacher H_1 used a problem involving the number of chirps a cricket makes and the temperature at the time. She was not only aware of many methods of solution to the problem, but expected the students to supply as many alternatives as they could. When some students had difficulty with using a formula solution to the problem the teacher consoled them with the observation that 'you can get into a bit of a mess of algebra' trying to use the method and steered them into the use of pattern-seeking methods. She was not bound to formula methods in solving the problems.

Use of Humour

With Teacher L_1 obvious incidences of humour took the form of playful banter between the teacher and students. The teacher smiled freely while helping students on a one-to-one basis while walking around the classroom. This pattern continued through both lessons.

There was a friendly, low-key rapport between the Teacher L_2 and her pupils. While there is a mildly mocking tone to the humour, it was well intended and well received by the pupils. Friendly teasing both between students and between teacher and students was a feature during the lessons. Even the teacher's admonition to an inattentive student towards the end of the second lesson, with a quiet aside of 'Are you going to join us?' was done in good fun.

Teacher H_1 generated humour by suggesting the unexpected, particularly in the first lesson. An example of the relationship between the number of chirps the cricket makes and the temperature at the time was reworked by the teacher in her suggestion that the data may have come from a science experiment in which a Bunsen burner was placed under the cricket to increase the temperature so that the scientist could note any changes in the number of chirps.

For Teacher H_2 the occurrence of humour was subtle during the first lesson with year 9. During the second lesson humour was not particularly evident. In the year 10 class this pattern was reversed.

Use of Pupil Effort

All the teachers made significant use of pupil efforts in the structure of both of their lessons. The efforts of Teacher L_1 and Teacher L_2 illustrate a typical pattern. In the first lesson, Teacher L_1 used pupil contributions to develop an understanding of line symmetry. On being asked for a meaning of symmetry, one student started an explanation that involved some hand waving and the teacher instructed the student to come up to the board to draw a figure if that was going to help her explanation. For

Teacher L_2 pupil contributions very much determined the pace, content and direction of the first lesson. Responses to the request to 'think of three facts about enlargement you could tell someone who didn't know anything about it' were written on the board using the pupils' words. The teacher was very interested in having the pupils find out for themselves the consequences of varying scale factors and centres of enlargement. At each stage responses from the floor were encouraged to describe the consequences of each of these manoeuvres. Pupil involvement was also evident in the assessment of their efforts. Both peer assessment and self assessment were used.

In the second lesson all teachers used solutions provided by the students, often asking students to come to the board to explain their reasoning.

Use of 'Real World' Examples

In all the second lessons observed, virtually all of the problems that were given to the students were set in a real world context. For the first set of lessons, the different topics chosen made it less likely that this would occur. As it turned out only Teacher L_1 made no use of real world examples.

Discussion and Conclusions

No attempt is made here to quantify the teaching visuality of the participants in the fashion of Presmeg (1985). Instead trends are identified and the possible reasons for them discussed. At this point several of the episodes in the interviews with the teachers become useful in explaining the motivation behind some of the actions of the teachers.

There were several aspects of teaching visuality that the four teachers appeared to have in common. All the teachers, regardless of their mathematical visuality, were supportive of visual interpretations to the problems in the second lesson, encouraging students to draw diagrams. They all made much use of pupil effort during the lessons, often encouraging them to use their intuition and accepting, almost expecting, several methods of solution to problems.

This common behaviour appeared to stem from a reflective approach that they had with respect to their teaching. In support of a visual method, Teacher L_2 noted that her experience led her to believe that students didn't like an algebraic method, so in the class she maintained that she 'went from what the kids did. . .I tended to try and picture it so they can see it more clearly than they can see it when it's in algebra.' On the other hand, Teacher H_2 was able to reflect on what worked for her and use it in her classroom: 'A diagram to me helps to summarise and make sure that I understand the problem and because I find it easy that way I encourage them to summarise it that way.'

This reflective approach and an exposure to alternative methods in teacher training courses seems to account for the consistent use of concrete material by Teacher L_1. In both lessons she planned for students to be engaged in some physical activity that used materials as an aid to the students' understanding of the lesson. When asked in the interview why she encouraged students to use manipulatives to help solve the problems, Teacher L_1 replied: 'Because I think it's a good problem solving strategy and I think it's probably something that I think's quite powerful but . . . it's only something that I've really been exposed to in recent times through courses and things like that. People have sort of given you a problem and shown how you can solve it using manipulatives and I think, yeah, it's so powerful ...I've missed out on that in earlier times . . .'

While there was much that was common in the teaching visuality of the teachers, there were differences. These did not appear to relate to the mathematical visuality of the teachers. It had already been noted that Teacher L_1, the lowest scoring mathematical visualiser, was the principal user of concrete materials. Additionally, the use of gesture was as likely to be observed with teachers of high mathematical visuality as it was with the teachers of low mathematical visuality. Teacher L_1 and Teacher H_1 were the most consistent in their use of gesture in the classroom, displaying some form of gesturing in both lessons. Both Teacher L_2 and Teacher H_2 were relatively reserved in their use of gesture.

The existence of a weak statistical relationship that Presmeg (1985) found in her sample appears unlikely with this sample of teachers. All four teachers displayed common characteristics of teaching visuality, while aspects that were not common were as likely to be observed with the teacher of low mathematical visuality as with the teacher of high visuality. While Presmeg (1985) observed that in her case study sample of thirteen teachers 'no teacher of low mathematical visuality was found to be a visual teacher' (p.269), in this research Teacher L_1 appears to fit the category of a teacher with low mathematical visuality who has high teaching visuality. With respect to the Problem Solving Questionnaire she confirmed her low mathematical visuality when she explained her solution strategy as: 'Most of the time my initial reaction is to try and go to an algebraic expression . . . you see, I've got my higher level algebraic skills. . .' In her teaching however she does much to encourage the use of visual aspects, being the most consistent of the four teachers in the use of concrete materials, while also employing spatially significant gestures in her communication with students. Diagrammatic approaches to solutions are accepted and encouraged even if they are not the teacher's personally preferred strategy. Her teaching is not 'rule bound' as she accepts alternative methods from students and encourages them to use their intuition.

Of interest is the attitude towards the use of visual methods that teachers displayed in the interviews. Teacher H_1 appeared particularly to see the drawing of a diagram as a developmental stage that people grow through 'like when you are starting with playing with manipulatives and then you can abstract it to drawings of them.' When noting that she still approached her problems from a diagram rather than immediately from an algebraic approach she responded: 'Maybe I've got another level I can go.' Later, in explaining her approach to the solution of a problem she said: 'I've got a method: my method is draw a diagram, put the numbers on the diagram, find the equation and solve it. That's my method. . . Obviously some of your higher-up people are missing the 'draw a diagram' stage.' The implication from this teacher with high mathematical visuality and high teaching visuality appears to be that capable mathematicians don't need diagrams, an attitude that Dreyfus (1994) contends may lead to a reluctance in students to continue to use visual arguments. Teacher attitude towards the use of visual methods in mathematics is a possible field of further research.

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