Teachers interpreting algebra: Teachers' views about the nature of algebra

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This paper looks at the different ways teachers interpreted and represented algebra for their students. In this study, teachers are viewed as mediating between the student and the mathematics content. The teachers in this study differed widely in their epistemological and pedagogical beliefs about algebra but tended to teach similar content. The results of this study suggest that teachers' choices are a consequence of particular beliefs about algebra and that these beliefs are attributable to cultural influences rather than academic considerations.

Introduction

The research reported in this paper evolved out of a concern that the choices that teachers make with respect to teaching algebra often do not relate to teachers' epistemological and pedagogical knowledge of algebra. Given the emphasis placed on algebra in the Australian mathematics curriculum (both nationally and at the state level), teachers' knowledge of algebra and the strategies they employ to teach it are fundamental to the realisation of national and state curricular goals in mathematics. The provision of appropriate professional development support for teachers, whether pre-service or in-service, must be based on an understanding of how teachers themselves conceptualise algebra, of their pedagogical knowledge in relation to the teaching of algebra, and of the bases for the instructional decisions they make in interpreting algebra content for their students. In this research, the pedagogical content knowledge related to the teaching of algebra was investigated in significant detail for a small number of teachers in a single rural secondary school.

The conception of the teacher as mediating between the student and the content offers a powerful perspective on teaching practice. According to the writings of Vygotsky (1962, 1978) cognition is a mediated activity. The vehicles by which the representations of cultural meanings are transferred to the learner include such social constructions as linguistic signs and discourse, orthographies, and numeration systems. For Vygotsky the cognitive development of the individual is possible only within the collective practices of the social group.

This study focused on teachers' pedagogical content knowledge with respect to algebra. Pedagogical content knowledge in relation to mathematics has been described and discussed by Shulman (1987) and Kieran (1992). With the work of these researchers in mind, this study sought to examined the terms teachers employed to describe the content, structure and nature of algebra and to describe teachers' pedagogical content knowledge and teachers' personal beliefs about algebra.

Pedagogical content knowledge

Shulman (1987) emphasised on several occasions that the teacher who is best able to successfully orchestrate a range of learning opportunities with a mixed interest and ability group of students is the one who has a deep knowledge of the topic, being able to see its many relationships to other topics and relationships of the elements within the topic.

The content, Shulman (1987) suggests, needs to be considered by teachers from the perspective of, for example, what is most important for students to know about. With respect to algebra this means that teachers need to understand what algebra is and how it is different from other forms of mathematics. Teachers need to know what knowledge needs to precede other knowledge within the domain of algebra. Teachers need to understand what students are able to comprehend, that is, what concepts are conceptually more difficult than others, the nature of these concepts and how they relate to each other. Then teachers also need to have knowledge of different representations by which ideas might be presented to students so that students are sufficiently interested and willing to entertain them.

Algebra in the curriculum

Algebra considered from the perspective of the curriculum can usefully be described in terms of what it might look like and what students might be doing. Kieran has accumulated a substantial body of research and theoretical writing on the nature of algebra as content to be taught and learned. Implicit in the writing of Kieran (1994) is that one of the most important conceptions of algebra for teachers, students and users of algebra is making the link between conceptualising the nature of algebra as a way of viewing phenomena (identifying mathematical problems including problems with variables and using the idea of a function) and expressing these in symbolic form (generalising and modelling) and as a process of manipulating these symbolic expressions in order to solve the identified problem. However, writers such as Wheeler (1997) say that algebra described for the curriculum is usually described in terms of algebraic strategies and the concepts that link the strategies to a purpose such as solving equations or graphing a function.

Anticipating possible student difficulties

A key aspect of a teacher's pedagogical content knowledge is an awareness by the teacher of likely student difficulties with particular aspects of algebra and knowledge of teaching strategies by which these might be prevented or overcome.

The difficulties with algebra are often couched in terms of the complexity of thinking involved when utilising algebraic structures and formulating mental models (MacGregor and Stacey, 1995). If we consider algebraic thinking, we need to consider more than just activities such as codifying, manipulating, and proving but also "the interactions and distinctions between formulating the problem equations and manipulating them to find the problems solutions" (Bell 1997, p. 151).

Algebra is like natural language in that there can be different denotations with the same sense and one denotation that can take on more than one meaning. Students will often have difficulty with this flexible interpretative nature of algebra and tend to treat expressions as static pieces of non-mathematical information (Mestre 1988, MacGregor & Stacey 1994c). According to Arzarello, Bazzini and Chiappini (1994, p 1-138), one obstacle to using algebra as a thinking tool concerns the fact that algebra does not use the same "natural" way of thinking as do familiar number problems.

The essential point here is that a significant aspect of a teacher's pedagogical content knowledge resides in the teacher's understanding of the difficulties that students are likely to experience and the strategies by which these might be corrected or prevented. This requires of a teacher that she have knowledge of the content of algebra within the domain of mathematics, of the characteristics of learners and learning as these relate to algebra, and of the representational forms and instructional techniques whereby students' learning of algebra might be promoted. This paper reports a study into the form taken by this pedagogical content knowledge for a small community of secondary mathematics teachers.

Methodology

A study of five mathematics teachers of the classes Years 7, 8, 9 and 10 (1995-1997) was carried out to determine empirically, by observation, questionnaire and interview methods, what each Teacher believed about algebra and teaching algebra. Data was collected over two years to validate information gathered because the researcher recognised that a teacher's expressed beliefs might change from one year to the next in accordance with the unique relationship that develops between the teacher his or her classes and changes in curriculum initiatives such as the *Curriculum and Standards Frameworks* (CSF) (BOS, 1995. 1997).

The following methods were used in this Study:-

- Questionnaire methods both oral and written
- Thinking aloud while engaging in problem solving
- Clinical interviews, both structured and interactive
- Naturalistic observations

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It was felt that each of these methods was important in the overall design of the research which took into account what often appeared to be conflicting statements and observations of teachers. Comparisons of the diverse data were then able to be made to enable valid conclusions to be drawn.

Interviews were of two types, "think aloud" and "clinical interview". The "think aloud" method involved the interviewer only taking part to provide a stimulus such as posing a problem and occasionally encouraging the subject to continue with their comments about the nature and importance of algebra. The "think aloud" method required the subject to say everything that comes into his or her head while, for example, solving a challenging problem. The "clinical interview" method involved considerable interaction between the interviewer and researcher. The study design took into account the need to inquire thoroughly into the teachers' thinking about algebra. A fixed set of questions was used as the basis of the interview but additional questions sought to clarify, give examples, provide embellishment or additional information. The goal of the interactive clinical interview was for the interviewer to obtain increasingly more specific information about what the teacher believed and why they made the pedagogical decisions that were observed.

Teachers had great difficulty, when interviewed, in describing such things as preferred mode of working and relating this to teaching practice. However, this particular relationship was central to the study so it was important to support clinical interview data with data from other sources. These included observing teachers working out algebraic problems, listening to the way the teachers explained algebra to students and to the researcher, and reflecting upon the teachers' written solutions to problems and teachers' written and verbal comments about written solutions of problems solved by others. Naturalistic observations were made to observe teachers teaching algebra to ascertain the type of language, illustrations and other mediators teachers used to explain algebra to students. During these observations, field notes were kept along with audio and video recording of teachers' and students' behaviour as it occurred 'spontaneously'. Teachers' behaviours were analysed with respect to the extent that they implied beliefs about algebra and teaching algebra.

Questionnaires were used in the study to collect initial information about what algebra was taught in the Years 7, 8, 9 and 10 classrooms along with what teachers regarded to be algebra content and appropriate ways of solving algebra related problems

Results and Discussion

This study was carried out in a small rural community college (catering for grades P to 12). A consequence of the small size of the school is that all teachers are mixed method teachers, teaching other subjects as well as mathematics. Over the period of the study, all secondary mathematics teachers (other than the researcher - Menzel) were included in the research.

To make the structure of the report as transparent as possible and consistent with the scale of this paper, results have been summarised as bullet points, with some brief elaboration of each point. The implications of each result are discussed briefly as they arise.

- Teachers' remarks about the importance of algebra for school mathematics fell into four broad categories:
  - The students' need for algebra for VCE studies;
  - Algebra as a useful alternative for solving problems;
  - Algebra as an important component in student's cultural education; and
  - Algebra as an important activity for developing thinking skills.

Three teachers explained that being able to do algebra was an important aspect of students' cultural education. They believed that our society expected students to study algebra. In the words of one teacher: "...I regard it [learning algebra] as general knowledge. ... It is probably important to teach all students algebra even though they will always have trouble with the concepts."

- Teachers' descriptions of algebra fell into two broad categories, those that referred to the curriculum and those that made specific reference to the epistemology of algebra.

Teachers' descriptions of the nature and function of algebra illustrated wide variation. Teachers' descriptions of algebra content tended to be in terms of algebra curriculum activities, concepts and skills, procedures and/or teacher action when describing functions, equations and generality but in terms of epistemology when describing the symbolic nature of algebra. In the case of the content of Algebra, the teachers were largely in agreement. Five terms used by all teachers to describe algebra were: Solving equations, finding relationships, symbols, factorising and using rules (such as for factorising quadratics). Possibly this was because the available curriculum documents described material in terms of common topics, and the teachers largely accepted this.

Teachers generally described the content of algebra in terms used to name units of work. For instance one teacher's concept map yielded the following terms: Graphs of quadratics, symbols & abstract information, expanding, simplifying & factorising, straight lines and equations, equations & functions, and spreadsheets, experimenting, understanding, memorising. This corresponded closely to the broad topic headings she used to summarise her algebra course. Her broad description of her Year nine algebra course was:-

symbolising and abstracting
first degree equations, expansion & simplification and
quadratics and line graphs

While actual algebra content conformed to the CSF, the terms by which the teachers chose to express this content revealed some interesting differences in the way they viewed algebra.

Each teacher said that algebraic activity can be identified by its symbolic form but differed in their emphasis on this as a distinguishing characteristic of algebra. The teachers generally believed that algebra is being used when one thinks about how to solve mathematical problems involving variables or unknowns even before any formal writing takes place.

Not all teachers believed that algebraic activities need to involve written symbols. Three of the teachers believed the use of formal symbolic notation is very important for effective algebraic solutions, while the other two teachers believed that effective use can be made of algebraic concepts by utilising students' understanding of algebraic procedures and using personal writing styles.

The use of verbs in both concept maps and interviews about the content of algebra suggested a conception of algebra in terms of skills while the use of nouns suggested a focus on broad concepts. The teachers made frequent use of verbs expanding, simplifying & factorising to describe a specific focus when considering teaching equations and generality. They used nouns to describe functions implying that related algebraic notions were not just carrying out procedures but also included broad concepts such as the notion of variable and understanding relationships. The teachers' descriptions of algebra as an approach to solving problems was usually in reference to the nature of algebra but did not feature highly when they described teaching algebra.

- Teachers differed in the way they viewed Generality.

All Teachers' made some reference to "abstract notations" and "symbols" in their concept maps and some reference to ideas of expressing relationships. One teacher used key words in her concept map of algebra that described Generality as "memorising - expanding, simplifying & factorising techniques ... but also understanding relationships and applying this to real data". The teacher's list specified implicitly her regard for the importance of the expression of empirical data in general terms, and describing relationships. Teachers all

made several references to ideas of generalising being an inherent part of what one does when doing algebra.

Differences in the teachers' views about Generality tended to be epistemological rather than based on curriculum content. For instance, the teachers were divided on epistemological grounds in their opinion as to whether verbal description of generalizable mathematical relationships should be regarded as algebraic activity. Teachers did not dispute the importance of verbal activity of this type but only whether such activity might be regarded as algebraic.

Teachers agreed that verbal descriptions should be regarded as an important activity to teach students to write an algebraic expression. However, for one teacher the verbal summation of the problem was seen as preceding the algebraic activity of symbolising as opposed to being an algebraic activity in and of itself while other teachers commented that verbal generalisations about numeric relationships constituted algebraic activity. Generally the teachers regarded generalising as having an integrative role for working with functions or equations.

- Teachers emphasised procedural knowledge for the study of algebraic equations

The idea of "Equations" as an algebraic concept was regarded by all teachers to be of significant importance for the teaching of algebra. The teachers' descriptions of the notion of "equations" centred on procedures such as factorising and backtracking. Teachers focused on curriculum units and emphasised working with formulas and skills such as transposing as important algebraic activity. The teachers agreed that constructing equations was an important aspect of algebra for solving many types of problems. The teachers were divided in their views about the legitimacy of substitution into equations as an algebraic activity. This difference is representative of some of the epistemological differences between teachers. For the purpose of teaching, all teachers endorsed the importance of specific procedural knowledge and solving equations.

- The concept of "Functions" was indicated by all teachers to be important for mathematics, if a lower priority for teaching.

There was a strong suggestion from the teachers that understanding the theory of functions lies in the domain of mathematics while application of functions is in the domain of science and other applied disciplines. This came across strongly when interviewing teachers. Such epistemological distinctions offer key insights into the form in which these teachers compartmentalised algebraic knowledge. Interviews with teachers also indicated that they regarded graphing techniques to be the most important aspect of functions for students to know about. The teachers' statements about the place of functions in the curriculum utilised terms expressing skills, such as sketching graphs from their equations, as opposed to understanding general concepts such as the concept of variable. For instance, one teacher spoke about functions in the following way:

> This [functions] is a simple procedure of taking an equation, making up a table of values and plotting these values. I personally don't see this as a difficult procedure but I guess there may be some students who do struggle with this concept. ... Drawing and reading graphs is not difficult"

This teacher's comments emphasised a view he shared with other teachers that his responsibility was in the teaching of procedural rather than relational aspects of functions. Teachers endorsed the importance conceptual knowledge involved in understanding functions but centred on understanding specific skills rather than broad concepts such as "variable" .

- Teachers focused on different concepts and skills

The teachers categorised the algebraic content in two sets of 14 questions differently. The teachers' focus on different aspects of algebra suggested that each teacher might have a different way of approaching a particular mathematical task. For example, the teachers' differed in their classification of the following task:

Three containers, each identical in size and shape, contain three different quantities of liquid. The first is twice the volume of the second and the same as the third. If the total volume is five litres, how much liquid is in each of the three containers?

The teachers' responses suggested three different views about the essentially algebraic content in the task:

- working with symbolic representations;
- interpreting a situation in symbol form; and,
- operating on algebraic objects.

These differences occurred even though the teachers uniformly suggested that the task essentially involved concepts related to equation-solving, such as 'equality', 'symbolic form' and 'mathematical operations'.

Examination of teachers' responses to the "three container" problem above indicated that two of the teachers' believed symbolising to be a significant or important skill required to solve the problem with one of these two teachers believing that operating with algebraic objects was also important. A third teacher believed that interpreting the problem was the most important skill. This, he explained, was because students will usually experience little difficulty finding a solution having once interpreted the original information correctly. A fourth teacher believed that setting up an equation from an understanding of the relationships inherent in the problem was a significant algebraic activity but that manipulating the equations was the most significant algebraic activity involved in completing the "three container" problem above. The fifth teacher believed that logical reasoning could be used to solve the "three container" problem above and that an algebraic solution would only be evident if an equation was constructed and solved in some formal way.

While all teachers agreed that the "three container" problem above involved primarily the solution of equations, each teacher regarded the problem from a different perspective. Each perspective was related to the way these teachers perceived the problem for the purpose of teaching algebra and therefore was described by teachers in light of the skills they believed the problem might best serve to illustrate.

- Algebra was identified by some teachers as a way of thinking.

Whether a problem involved algebra or not was decided by four teachers partly on the basis of their identification of the "algebraic thinking" they regarded to be involved in the problem. All the teachers commented that algebraic thinking could be described in some form. However, whether algebraic thinking could or should be taught depended upon how it was conceived by the teachers.

One teacher's comment was: "You have logic statements and you have algebraic statements as well ... algebraic thinking is simply using the algebra rules and incorporates a knowledge of the development and application of these rules". A different teacher saw algebraic thinking as general ability to process any mathematically abstract information or ideas. A third teacher restricted her description of algebraic thinking to the "... use [of] algebra for abstract thinking needed to solve many problems in maths". The contrasting views being expressed present algebraic thinking as the understanding required for the manipulation of algebraic symbols or a more general form of abstract mathematical thinking related to identifying fundamental abstractions relevant to a particular problem.

In addition, all the teachers shared a view that one component of algebraic thinking was being able to think logically in order to be able to complete algebraic tasks. Teachers emphasised that the logical thinking needed to have a sound theoretical base that included at least knowing algebraic procedures.

The teachers' descriptions of algebraic thinking differed in emphasis more than in content. The teachers' opinions were rationalised in terms of pedagogy rather than epistemology.

Conclusions

During the study the teachers investigated tended to describe algebra in curricular terms rather than engaging in personal theorising about the nature of algebra. The teachers described algebra concepts and skills in terms of topics such as factorising, solving equations and writing algebraic expressions to represent a mathematical problem or situation. Algebra was described as a mental process involving logical reasoning. This form of logical reasoning was associated by teachers with understanding the logical sense of algebraic procedures and being able to use these procedures. The teachers' conceptions of algebra differed in emphasis from each other.

The teachers' descriptions of algebra included a view that algebra provided an approach to solving problems. Three of the teachers suggested that algebra should not always take a predominant role in the classroom when solving mathematical problems with numerical methods being regarded with equal importance. However, one teacher expressed a view that algebraic methods would often prevail in problem solving because of their efficiency. However, for all teachers algebra provided a structure or framework of concepts from which to approach particular types of mathematical problems.

The teachers did not endorse a dominant role for algebra in the school curriculum to the same extent. Differences among teachers were based on epistemological considerations and concerned opinions about whether numerical methods were more efficient. These differences of opinion related to the teachers' personal preferences with regard to methods of solving problems.

This research has documented the diversity of influences and consequent beliefs contributing to the way in which teachers interpret the nature and purpose of algebra for their students. It is not surprising that teachers' conceptions of algebra should give priority to curricular matters in preference to epistemology. What is noteworthy, however, is the degree of difference in significance that individual teachers attach to particular algebraic concepts and skills. These differences raise the question of how well-equipped teachers of mathematics must be in order to make decisions about instructional priorities. This study suggests that some teachers present algebra as something you do, rather than emphasising the central ideas and concepts that characterise algebra. Teachers adopting the former approach are more likely to give students a limited, utilitarian conception of algebra. Such a conception lacks generality and may seriously restrict students' recognition of connections between this knowledge and both 'real world' situations, in which it might be applied, and with the broad domain of mathematics, in which it should find its meaning.

These findings, albeit of a small group of teachers, suggest that 'pedagogical content knowledge' offers a useful window into the thinking underlying the practices and beliefs of mathematics teachers. Should the findings of this study be typical of conceptions of algebra held by the general mathematics teaching community, the implications for teacher professional development, inservice and preservice training and new curriculum initiatives could be profound. Further analysis of this data, may reveal specific steps by which these concerns about teachers' mediation between student and content might be addressed.

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