"This is a funny game - you can't say who's going to win!": Three case studies of children's probabilistic thinking

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Three case studies of children are used to illustrate the variety of strategies employed by children when asked to make probability judgments in several different game contexts. The children's responses ranged from idiosyncratic and intuitive reactions to the deliberate application of proportional reasoning. It was found that certain combinations of variables in the task designs stimulated different mathematical thinking.

Introduction

In an exploratory study of children's probabilistic thinking, task based interviews with 74 children, aged from 5 to 12 years, were conducted. This paper reports the information gathered about three of the children; Jessica (5 years and 11 months), Jason, (9 years and 11 months), and Jack (11 years and 8 months). The children were selected from the 74 simply because their interview responses covered a range of the types of thinking observed in the children, and were also representative of the type of responses in their age groups. These children were considered by their teachers to be of average mathematical ability.

The key research questions for the exploratory study were:

1. What strategies do children utilise for making judgments in different types of probability tasks?
2. What is the relationship between these strategies and the type of probability task?
3. Can the children's responses be classified into the expected developmental stages of non-probabilistic thinking, estimation of probability and quantification of probability?

In order to gain information which might provide some answers to these questions a set of interview tasks, in the form of games, were designed around the following variables: the type of random generator to used (*numerical* - objects in a jar or *spatial* - spinners); the structure of the sample space involved (such as the number of colours used and the relative amounts of each; the nature of the comparison being requested (such as within the one sample space or between sample spaces); and the type of responses expected from the child (usually a choice from two or more items and a reason for the choice). The identification of these variables, and the design of the tasks was influenced by the work of a number of people, including; Fischbein and his colleagues (1970), Hoemann & Ross (1971), J.Truran (1994), and K.Truran (1995). A range of specific concepts associated with probability were covered by these tasks. These concepts are listed in Table 1, together with a summary of the structure of each interview task.

The Interview Tasks

Task 1: Bears in a Box : Four small coloured bears (3 of one colour, 1 of another colour) are placed in a box. The child is asked to say which colour is most likely to be drawn out, and why the choice has been made. The child draws out a bear and then replaces it. This is repeated 5 times, with a display kept of each outcome.

Task 2: Non-replacement : This is similar to the first task except the bear is not replaced after each draw.

Task 3: Racing Cars : Four spinners of differing construction are used to play a game where coloured discs are moved along a race track. The child is asked to make various choices in regards to the most likely winner using a particular spinner, or the best spinner to use to get a certain result. Reasons are sought for each choice.

Task 4: Transfer : The child is asked to place coloured bears in the box to replicate the colour ratios of the spinners from the Racing Car game and to explain how they decided on the number of bears of each colour.

Task 5: Proportions : Two jars of various mixtures of red and yellow bears are displayed to the child, who is asked to choose the jar that will give the better chance of drawing out a red. The coloured bears are lined up outside the jars to facilitate the child's choice and explanation.

Table 1: Summary of Analysis of Interview Tasks

Interview Task	**Type of Random Generator**	**Structure of Sample Space**	**Nature of Comparison**	**Type of Response**	**Specific Probability Concepts**
Bears in a Box	numerical	2 colours 3:1	Intra-sample space	Choice of colour Reason	Sample space - constant Randomness More likely
Non-replacement	numerical	2 colours Initially 3:1 then changing	Intra-sample space	Choice of colour Reason	Sample space - changing (Simple conditional probability) Randomness More likely Equally likely
Racing Cars	spatial	4 or 3 colours A 1:1:1:1 B 4:2:1:1 C 3:1:2:2 D 3:1:4:0	Intra & Inter-sample space Same type of RG	Choice of colour Choice of sample space Reason	Sample space & Randomness More likely/Most likely Less likely/Least likely Equally likely Impossible/certain Spatial proportions
Transfer	numerical & spatial	4 colours A 1:1:1:1 B 4:2:1:1	Inter-sample space Different types of RG	Choice of numbers of items Reason	Sample space & Randomness More likely/Most likely Less likely/Least likely Equally likely Impossible/certain Proportions, spatial & numerical Quantification of chance
Proportions	numerical	2 colours 1:4 & 3:4 2:4 & 4:8 1:4 & 3:6 1:3 & 2:8	Inter-sample space Same types of RG	Choice of sample space Reason	Sample space & Randomness More likely Proportions, numerical Quantification of chance

Responses to the Tasks and Discussion

Task 1: Bears in a Box

Jessica: (5.11yrs) Jessica placed 3 Red(R) and 1 Yellow (Y) bears in the box, but chose Y as the more likely outcome for the first draw, saying "*'cause I like it*". Before the third draw (after drawing a Y then a R) Jessica chose R and gave the reason, "*I like it very much and that's my second, um third best colour*". After drawing a Y and a R, her third response was to choose Y again, saying "I like yellow". The final two draws resulted in a R and a Y. Jessica then predicted that another six draws would produce different results. She explained it would be different by saying, "*There's three red and three yellow, so that's six - red, red, yellow, yellow, red, yellow*". Jessica's reply to the question about why the box shaken before each draw was, "*To mix them up*".

Jason: (9.11yrs) Jason chose to put 3 green (G) and 1 red (R) into the box. When asked to say which colour he thought he was more likely to draw out, he said, "*Green,* *'cause there's more of green"*. His first draw resulted in R, but his second draw produced a G. When asked to again predict, he responded, *"Green, 'cause there's more of it"*. Following two more draws of R then G, the third question elicited the same response. The final two draws were R then G.

Jason said that he thought another six draws would produce a different result, *"'cause you might pick different ones - maybe 4 Green and 2 Red"*. Six more draws gave the sequence RRRGGR, to which he exclaimed, *"Wrong way around!"* Jason's answer in regards to the box shaking was, *"Stirring them around so you get different ones"*

Jack: (11.8yrs) Jack's selection was 3 Blue and 1 Red, and he correctly specified Blue as the more likely outcome of each draw, with comments of: *"There's three blue and only one red"*, *"Still blue. It's just chance. There's more chance to get one of the blue than the one red"'*, and *"Blue - same reason"*. The results of his draws were RRBBRB. He decided that six more draws would produce different results and said, *"Mainly just get blues"*. The results were RBRBBB. When asked why the box was shaken Jack responded, *"So that you wouldn't know were they were in the same position and so keep pulling the same one out"*.

Jessica's responses indicate non-probabilistic reasoning. She does not appear to realise the connection between the structure of the sample space and the likelihood of various outcomes. Although there is some acknowledgment of the concept of randomness, the understanding appears weak. Although Jessica said the next six draws would be different she seemed to believe there would still be three Red and three Yellow drawn.

Jason's responses were classified as being at the *Estimation* level because they clearly utilised knowledge of the structure of the sample space, were not influenced by the outcomes of each draw, but did not involve any quantification. He showed an understanding of the concept or *more likely* and also of the idea of *randomness*. The data sequence generated by the draws did not influence his predictions which suggests he has a sense of the independence of each draw.

Jack's responses suggest a willingness to use numbers to describe likelihood. He exhibited a strong understanding of the way in which knowledge of the structure of the sample space can be used to determine likelihood, yet also understands that the outcome of each draw remains essentially unpredictable.

Task 2: Non-replacement

Jessica: With three Red and one Yellow in the box to start with, Jessica predicted Y each time saying, *"Because it's a good colour"'* and *"Sometimes it's on flowers"*. The equally likely situation did not arise, even though the game was repeated twice. However, it was apparent that Jessica was able to keep mental track of the contents of the box after each draw.

Jason : Jason again placed three Green and one Red bears into the box, and again stated that G was more likely to be drawn, *"'cause there's more of them."*. A Gr was drawn, but not replaced. Jason's prediction was G for the next draw, *"'cause there's still more of them"*. A G was once again drawn. When asked which colour he thought more likely to drawn next he replied, *" It's a lucky guess."* ("Why?", asked the interviewer). *"They're the same in there. I might get the Red"*.

Jack: With 3 Blue and 1 Red in the box, Jack chose B as the more likely outcome, and a B was drawn. Before the next draw Jack again chose B, saying *"There's at least one more chance to get blue than red"*. A R was drawn next. Jack's reply to the final question was, *"It's even, there's one of each"*.

Jessica paid little attention to the structure of the sample space and gave no indication of an awareness of the differing likelihoods of outcomes.

Jason had no difficulty in keeping mental track of the contents of the box, that is, the changing sample space, and so his responses implied an understanding of very basic conditional probability. His last response indicates recognition of equal likelihood.

Jack acknowledged the changes in the sample space and the impact on the likelihood of the possible outcomes. He also was able to recognise equal likelihood.

Task 3: Racing Cars

Jessica: After playing a game with the Spinner A (all colours equally likely) to establish the context of the task, Jessica was shown Spinner C and asked to say what colour car would have the best chance of winning the race, playing with that spinner. Jessica responded incorrectly by saying, "*Green, because you won the last race with Green. This is a funny game - you can't say who's going to win!*". She lost the game.

Next, Jessica was told she must be the Red car, but could choose the spinner she thought would give her the best chance of winning the race. She was asked to select the spinner from the whole set of four spinners. She correctly chose Spinner D, saying, "*It's got the highest bit there* (pointing to the red half of the spinner)". She won the game played with this spinner.

Jessica was then asked to select, from the set of four, the spinner that should be used for a *fair* game; one in which each of the four colour cars would have the *same chance* of winning. She gave the reason, "*They're all got the same shape*". (No more games were actually played).

Next, Jessica was asked if there was a spinner that would make it *impossible* for the Green car to win the race. She chose correctly, with the explanation, *"I think it would be hardest for Green - it's got none!"*.

The next question required Jessica to identify the spinner that would give the Yellow car the *best chance* of winning the race. This was correctly chosen, with the statement, "*It's got the biggest yellow*". Jessica was quite certain that yellow would win and explained, "*Yes, because if I spin it will go there, there, there on yellow all the time (pointing to several locations on the yellow sector*".

Finally, Jessica was asked to identify the spinner that would make Red the *least likely* to win, that is, the best spinner to make Red lose the race. Jessica chose correctly, saying "*Because it's got the tiniest red*". She was also certain that Red would lose - simply saying, "*Yep!*".

Jason: Jason correctly chose the Blue car after looking at Spinner C, giving the explanation, "*'cause there's more space of Blue than the others - you might spin it. It's got more chance of landing there*". Jason then played the game as the Blue car and won the race.

Jason correctly chose Spinner B for the Red car to win with, saying "*It's got a full half. That one's* (Spinner D) *not quite half*". Jason played the game as the Red car and won again.

Jason correctly selected Spinner A as the *equally likely* spinner saying, "*'cause it's got the same amount of each one*".

Jason explained his choice of Spinner D for impossible-for-Green by saying, "*There's none on it - there's no green there.*"

He correctly chose Spinner D as best-for-Yellow because, *"It's got more - more yellow on there*". When asked if was certain that Yellow would win he responded, "*No - 'cause the other cars 've got a chance as well.*"

Jason picked Spinner C for Red-to-lose and explained, *"It's got the least red on it*". When asked if it would lose for certain, he said, "*No - it's still got a chance of winning.*"

Jack: Jack's reason for choosing the Blue car for Spinner C was "*It's got a large percentage of the circle - it's got more of a space where the arrow can land*".

Jack explained his choice of spinner for the Red car by saying, "*It's half Red and there's three others (colours) to make the other half*".

The equal chance spinner was correctly identified and Jack said, *"It's all split up into quarters - everyone's got the same percentage"*.

The choice of spinner for impossible-for-Green was justified by, *"It's got no Green on it"*.

Jack correctly chose the spinner most likely to produce a Yellow win, saying, *"It's got half of Yellow and there's two (colours) to make up the other half"*. When questioned about the certainty of the result of a game he said, *"It's not 100%. It has to be definite - it's got a 50/50 chance"*.

Jack explained the correct choice of the Red-to-lose spinner by saying, *"It's the lowest percentage of Red"*. When asked about the certainty of Red losing he replied, *"There is red on the card, if there was no red it would lose"*.

Jessica's first response relied on the result of the previous game rather than the structure of the sample space, and therefore indicated non-probabilistic reasoning. However, the rest of her decisions were based on the size of the colour sectors on the spinners, showing that she was able to make successful comparisons between sample spaces of this type. Her responses are categorised as *Estimation* strategies. Although Jessica noticed the unpredictability of the outcome of a game earlier, her belief in the certainty that her predicted outcomes would occur indicate an incomplete understanding of the notion of randomness.

Jason's series of responses indicate that he felt quite comfortable in making comparisons among several sample spaces, not just focusing on the outcomes of a single sample space. He demonstrated an understanding of a range of probability concepts, including *most likely, least likely, equal likelihood, impossibility* and *certainty.* Most of his responses were at the *Estimation* level, which was sufficient to clearly explain his thinking in terms of *spatial proportions*. However, in one response, he did successfully use fractions to make comparisons between two sample spaces, and therefore reasoned, in this particular situation, at the *Quantification* level.

Jack, like Jason, made the required comparisons both within and between sample spaces with ease and demonstrated an understanding of a range of probability concepts. However, Jack also showed confidence in using fractions as part of his reasoning, which placed most of his answers in the *Quantification* category. There is little doubt that Jack has a sound understanding of the concepts of certainty and impossibility in this context.

Task 4: Transfer

Jessica: In this task, Jessica was presented with the scenario that we wanted to play the Racing Car Game but had lost all the spinners. Could we put some Bears into the box, so that the colour drawn out would tell us which colour car to move forward? Jessica appeared to understand the nature of the task, and when asked to create a fair game she responded, *"All of them. I'd put 3 blue, 4 yellow, 2 red, 5 green. That's if you were Green. But 5 of each for fair"*. Her answer to the impossible-for-Green question was, *"Some of the other colour and none or one of Green"* .

Next, Jessica was shown Spinner B and asked to work out how many of each colour bear to put into the box to make the game 'work the same way' as the spinner would. Jessica lined up four bears along the edge of one of the sectors on the spinner - a radius of the circle. She placed four of each colour bear into the box and explained, *"I counted it up from down to up"*. When asked if the box would be likely to produce the same winner as the spinner Jessica said, *"I think I'll put some more extra reds in - so Red should win, but I don't care about the others - but Yellow looks like it should have a bit more."* The final contents of the box were 6 Red, 4 Yellow, 4 Blue, 4 Green.

Jason: Jason understood the context and did not hesitate in placing one bear of each colour (red, blue, green, yellow) into the box when asked to 'set up' a *fair* game. When asked if he could put some more bears into the box but still keep it fair, he said, *"Two - as long as they're the same."*

Jason was then asked what he would put into the box to play a game in which it was *impossible* for Green to win the race. He simply stated, "*No green*".

When asked to replicate Spinner B, Jason placed 4 Red, 2 Yellow, 1 Blue and 1 Green into the box, which correctly matched the ratio on which the spinner was designed. When asked to explain his thinking Jason said, *"Blue and green had the least so I put in one, and yellow had the second most so I put in two - and that one's* (Red) *got four"*. "Why?" *"Four of those* (Blue sector) *can fit in there"*.

Jack: Jack's initial response when asked to place bears in the box for a fair game was, "*Seven of each colour*". When asked if there was another amount that could be used he replied, "*You can put one of each, but you'd have to keep putting them back in*". Jason had at first assumed that the bears would not be replaced after each draw and so had put in one bear for each race-track space so that each colour car would have the opportunity to reach the finish line.

For the *impossible*-for-Green spinner Jack simply said, "*Don't put any Green in*".

To replicate Spinner B, Jack began by saying, "*Five red*", then paused for a while to think. "*I'm going to have to chop some in half. 5 red make up a half, 2 $^1/_2$ for yellow, 1 $^1/_4$ for blue and green*". The interviewer asked if changing the number of red bears would make it possible to actually put the right number of bears into the box. "*Probably could*", Jack replied. When asked why he chose to use five he explained, "*Scaling down from 100%, make it 10 bears go in* (to represent the total circle), *so half is 5*".

Jessica's responses in this task suggest the beginning of understanding in some probability concepts. She had successfully identified the *fair* spinner in the previous task, and now, after a shaky start, was able to construct her own *equal likelihood* sample space using number. Jessica's comprehension of the meaning of impossibility in relation to a sample space was once again shown to be incomplete. She thought that having only one Green in the box would still guarantee the Green to lose the race. In the final question, Jessica at first attempted a simple measurement strategy, but when she realised that this had not been successful, she made an incomplete attempt to relate the size of some sectors to the number of bears of that colour. Although her focus appeared to be on the winner of the game, there was an incomplete attempt at ordering the likelihood of each race outcome.

Jason was able to once again illustrate the notion of equal likelihood, this time in a way that showed his ability to work with proportions. He also demonstrated that his understanding of impossibility was not limited to the spatial random generator (spinner), but could be modelled with discrete items (bears). Although, in response to the final question, Jason did not use the language of doubling, halving or fractions, it is clear that he was 'unitising'. That is, he was using the smallest sector in comparison to the other sectors, in order to quantify their relative sizes - he was building ratios. Jason had little difficulty in comparing the two different types of random generators.

Jack's responses to the equal likelihood questions demonstrated his understanding of basic conditional probability because he was able to think through the consequences of non-replacement after draws, and apply this understanding to design an appropriate random generator for the game. Jack used a part/whole approach to quantify the size of the sectors on Spinner B. He began with a number to represent the whole circle, then used the fractions he estimated by looking at the spinner to determine the number of bears required for each sector. That is, he began with 10 for the whole circle - he could *see* that the red sector covered *half* the circle so he *halved* 10 to determine there should be 5 red bears in the box, and so on.

Task 5: Proportions

The final task was designed to further promote numerical thinking and hence provide the opportunity for the quantification of chance. Unlike the spinners, the jars of bears in this task could represent a wide range of ratios, with a variety of denominators. It must be remembered that there is a 50% chance of choosing the correct jar, and it is the reasoning behind the choice that provides the most useful information about the child's thinking.

Jessica: Game 1: Jessica was asked to choose between jars containing the following ratios; 1:4 and 3:4. The second jar was correctly chosen and the reason given was; *"The second one looks easier. There's 3 red and 1 Yellow".*

Game 2: The jars contained 2:4 and 4:8. Jessica incorrectly chose one jar (the second one) saying, *"It's got the most of reds".*

Game 3: When shown the 1:4 and 3:6 jars, the correct jar was chosen, with the explanation, *"There's about 3 reds and if I get 3 reds I'll win".*

Game 4: The jars contained 1:3 and 2:8. Jessica incorrectly chose the second jar, saying *"It's got 2 reds and if I get that I win another game".*

Jason: Game 1: Jason correctly chose the second jar, saying, *"This one would have more chance of getting reds 'cause it's got 2 more than the other jar, and yellow's would be the same".*

Game 2: Jason responded correctly. *"Um... Yellow's more likely...For red...they're both the same because there's double the amount of red in that jar, and there's double the amount* (of yellow) *in that jar".*

Game 3: Jason incorrectly stated, *"They're both the same because that one's* (1st jar) *got 3 more yellow* (than red), *and that one's* (2nd jar) *got 3 more yellow than red too".*

Game 4: Jason correctly chose the first jar, but not for the most appropriate reason, "*'cause this one's* (1st jar) *only got 3 more* (yellow than red) *and this jar's* (2nd jar) *got 6 more* (yellow than red)".

Jack: Jack chose the correct jar/s in each game. His reasons are listed below.

Game 1: (1:4 and 3:4 jars). *"This jar* (2nd jar)*would give the best chance at picking out red 'cause there's 3 red and in the other jar there's only one red".*

Game 2: (2:4 and 4:8 jars). *"They've both got half the percent of red of the yellow".*

Game 3: (1:4 and 3:6 jars) "*'Cause that's* (2nd jar) *half* (red half of yellow), *and that's* (1st jar) *only quarter."*

Game 4: (1:3 and 2:8 jars). *"This* (1st jar) *is one third and this* (2nd jar) *is one quarter".*

Jessica chose the second jar in every game and there is little evidence that she was able to make any numerical comparison between the two 'whole' sample spaces. There was a suggestion of the use of visual estimation when she says "It looks easier" and "It's got the most of reds", a strategy that was successful for her in the *Racing Car Task.* However, her main strategy was to choose the jar with the greater number reds in it, regardless of the number of yellows. In other words, she only used the *favourable outcome* part of the sample space in her reasoning, so there is no perception of ratio evident.

Jason used several different strategies in his decision making. In the first game he simply compared the quantity of the favourable colour (red) in each jar. This was appropriate because the number of yellow bears in each jar was the same. In the second game, Jason used doubling to explain the equivalence of the ratios contained in jars - not unlike the thinking he used to create *equally likely* ratios in the previous task. In the third game, Jason first compared the quantity of red and yellow within each jar to find the difference score for each jar, then compared these two scores. When he realised the differences were the same for both jars he concluded that each jar must present the same likelihood for red being drawn. This strategy is quite unlike any he had previously used. In the fourth game, it appears that Jason is again trying a subtractive strategy but incorrectly calculates one difference.

Jack once again applied his understanding of fractions to the comparison of sample spaces. He was able to state the ratio of red to yellow (in the form of a fraction) and then compare these in terms of the likelihood of drawing a red. His ability to use proportional thinking was highlighted by the recognition of equivalent fractions in Games 2 and 4.

Conclusion

The three children used a range of strategies in their decision making. Jessica's strategies were mostly non-probabilistic and her thinking idiosyncratic. She did, however, demonstrate some early understanding of some basic concepts that underlie probability. Jessica found it easier to make comparisons when the sample spaces were represented on the spinners. Jason's understanding of probability was more developed and he was able to use probability estimation strategies quite effectively, except in the final task. Jason's attempts to apply his emerging understanding of fractions and ratio indicate a readiness to learn new skills in this area. Jack's more sophisticated understanding of fractions and ratio equipped him to confidently find effective decision making strategies to suit each task, and enabled him to quantify probabilities. Both the older children illustrated the importance of developing doubling and halving skills, as these provided the foundation for several successful strategies. The final two tasks were clearly more challenging than the others, and accordingly, stimulated more complex mathematical thinking.

The differences in performance could be accounted for by developmental theory such as Piaget and Inhelder's (1951). However, the different levels of understanding can also be described in terms of the number and type of relationships the children were able to perceive between the variables in the tasks and their existing mathematical knowledge, such as used by Watson & Collis, 1994. None of the children had received any systematic instruction in probability; it not being a topic included in the curriculum. Therefore, the effectiveness of the strategies invented and applied by the children not only depended on their intuitive understanding of probability concepts and their abilities to work with fractions, ratio and proportion, but also on their ability to perceive the connections between the range mathematical elements involved in the tasks. The ways in which the task variables identified in this study (see Table 1) can be manipulated to stimulate various types of mathematical thinking in different age groups warrants further investigation.

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