An Overview of a Research-based Framework for Assessing and Teaching Early Number

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This paper provides an overview of a research-based framework for assessing and teaching in early number which has been used extensively in research and practice. The framework includes the following aspects: arithmetical strategies, forward and backward number word sequences, numeral identification, base-ten strategies, combining and partitioning, subitising and spatial patterns, temporal sequences, finger patterns, and quinary-based strategies. Five of these aspects are presented in tabular form as stages or levels. Also described are guiding principles for studying early number knowledge.

The purpose of this paper is to provide an overview of the Learning Framework in Number (LFIN), a research-based framework for the assessment and teaching of number in the early years of school. The LFIN was developed during 1996, for the NSW Department of School Education's Count Me In (CMI) project, and has been used extensively by all participants (including teachers, district consultants and researchers) in CMI and Count Me In Too (CMIT), a project subsequent to CMI. Since 1992, LFIN and its antecedent forms have constituted the theoretical framework for the Mathematics Recovery Program (Wright, 1994b; Wright, Cowper, Stanger & Stewart, 1995; Wright, Stanger, Cowper, & Dyson, 1996) and several empirical studies (eg Wright, 1994a). Thus the LFIN has been used extensively in research and practice. Finally, development of the LFIN drew extensively on recent research in early number learning.

Table 1
Organisation of LFIN.

Part A	A1 — Early Arithmetical Strategies
	A2 — Base-Ten Arithmetical Strategies
Part B	B1 —Forward Number Word Sequences (FNWSs)
	B2 —Backward Number Word Sequences (BNWSs)
	B3 — Numeral Identification
Part C	C1 — Combining and Partitioning
	C2 — Spatial Patterns and Subitising
	C3 — Temporal Sequences
	C4 — Finger Patterns
	C5 — Quinary-based Strategies
	C6 — Early Multiplication, Division and Fraction Knowledge

Organisation of the LFIN.

The LFIN is organised into three parts which span 11 aspects of early number learning (Table 1). All aspects are considered to be of potential significance when one is attempting to comprehensively understand and document a young child's number knowledge. For each aspect, the LFIN sets out descriptions of children's strategies or knowledge and a range of exemplary tasks or guidelines for assessment. Because of space limitations, detailed descriptions of these are not provided in this overview. Each of the five aspects in Parts A and B is presented in tabulated form as stages or levels. These tabulated forms — referred to as models — facilitate profiling of a child's early arithmetical knowledge. In four of the models the initial level (or stage) is labelled "Level 0 (or Stage 0)" and is also labelled "emergent". The description at Level 0 is typically in terms of the child not being able to perform at Level 1. Each of the 11 aspects can be regarded as viewpoints from which to observe young children's strategies in number and to document their number knowledge. These aspects should not be regarded as necessarily distinct from one another. Rather, they are potentially overlapping. Thus observations of two or more aspects may reveal similar features of a child's early number knowledge. As well, the extent of overlap in this sense, is likely to vary widely across young children.

Guiding Principles for Studying Early Number Knowledge

Development of the LFIN was informed by an extensive and on-going research program in early mathematics. This research program includes use of a distinctive approach to observing and documenting young children's arithmetical knowledge and strategies, which has been developed by the author and which has its origins in the constructivist research methodology developed by Steffe and colleagues (eg Steffe & Cobb, 1988). Outlined in the following section are some of the guiding principles and associated terminology of this approach. The use of this approach has provided an extensive empirical base for development of the LFIN. This base consists of videotaped records of individual interviews during which children attempt to solve mathematical tasks which are problematic for them.

Observation and assessment : Underlying the LFIN is a belief that, in early arithmetical learning it is very important to understand, observe and take account of children's knowledge and strategies. Children's early arithmetical knowledge varies greatly and their strategies are multifarious. Thus, across children, early arithmetical knowledge is characterised by both commonalities and diversity. It is insufficient to think that all children's early arithmetical knowledge develops along a common developmental path (Denvir & Brown, 1986). For example, one important factor in a particular child's developmental path it is believed, relates to the nature of the settings in which the child's prior learning has occurred. As well, children who may appear to an observer to be in the same setting or learning situation will construct the situation idiosyncratically and thus different kinds of learning are likely to occur. The child's process of constructing arithmetical knowledge can be thought of in terms of progression or advancement. Children reconstruct or modify their current strategies and doing so is nothing more or less than progression, advancement or learning. Given this, it is useful to consider the notion of the relative sophistication of children's strategies. For example, the child who has no means of working out nine plus three other than counting out nine counters from one, counting out three counters from one, and then counting all of the counters from one to 12, is using a less sophisticated strategy than the child who ignores the counters and says nine plus three is the same as ten plus two, and I know that is 12 without counting.

Settings, tasks, strategies and procedures : In the LFIN, the term "task" is used as the generic label for particular problems or questions presented to a child in the interview, while "setting" is used as the generic label for physical situations used by the teacher in posing a task. Examples of tasks are: given two covered collections, asking the child how many in all; asking the child to say the number words from "twenty" to "ten"; asking the child to hold up four fingers on each hand and say how many in all. Examples of settings are: a row of numeral cards, two piles of counters, a Tens Frame, a Hundreds Chart. A "strategy" is a particular method by which a child solves a task. Finally, a "procedure" is a part of a strategy, and strategies consist of one or more procedures. As well, specific procedures may be part of differing strategies, and specific procedures may be relevant to more than one of the aspects in the LFIN. For example, the procedure of raising fingers in turn to keep track of counting-on is relevant to the aspects of arithmetical strategies and fingers and finger patterns.

Presenting assessment tasks : A necessary part of determining a child's strategies is to present the child with tasks or situations which are problematic for her or him. In such situations the child's goal we assume, is to resolve the difficulty, ie to solve the problem as he or she constructs it. In doing so the child uses a current strategy or may construct a strategy which is novel for that child. It is not essential that the tasks presented to children be what adults might label as "real-world" or everyday situations.

For example, it can be just as useful and productive to pose a problem about numbers, number words or numerals per se, as it is to pose a problem about an everyday situation. Observations of children's solutions to a range of arithmetical problems lead to an understanding of the range, diversity and progression of children's strategies. With this knowledge one can determine a profile of arithmetical knowledge for individual children.

Eliciting strategies **:** A final but important point is that children frequently use strategies that are less sophisticated that those of which they are capable. This may happen for one or more of several reasons, eg (a) it may be easier and although it may take more time, this may not be of concern to the child; or (b) some feature of the child's thinking immediately prior to solving the current task may focus the child's attention on a less sophisticated strategy. Thus an important challenge for the teacher as observer and diagnostician is to attempt to elicit the child's most sophisticated strategy. This is crucial to gaining an adequate understanding of the child's current knowledge. Eliciting a strategy is not necessarily the same as asking the child to describe their strategy, after he or she has solved a problem. Close observation and informed reflection while interviewing are crucial to understanding a child's strategy. Asking a child to describe a strategy is sometimes useful but equally, can be off-putting for the child particularly if this becomes a routine directive. As well, when asked to describe a strategy just used, a child may unwittingly or intentionally describe a strategy different from the one used.

Aspects of the Learning Framework in Number

Following are descriptions of all but one (ie early multiplication, division and fraction knowledge) of the 11 aspects listed in Table 1, and each of the first five of these is presented in tabular form.

Table 2.
Model for Development of Early Arithmetical Strategies

Stage 0: Emergent Counting. Cannot count visible items. The child either does not know the number words or cannot coordinate the number words with items.

Stage 1: Perceptual Counting. Can count perceived items but not those in concealed collections. This may involve seeing, hearing or feeling items.

Stage 2: Figurative Counting. Can count concealed items but counting typically includes what adults might regard as redundant activity. For example, when presented with a collection partitioned into two parts (both screened), told how many in each part and asked how many counters in all, the child will count from "one" instead of counting on.

Stage 3: Initial Number Sequence. Child counts-on rather than counting from "one", to solve addition or missing addend tasks. The child may use a count-down-from strategy to solve removed items tasks (eg 17-3 as 16, 15, 14 - answer 14) but not count-down-to strategies to solve missing subtrahend tasks (eg 17-14 as 16, 15, 14 - answer 3).

Stage 4: Intermediate Number Sequence. The child counts-down-to to solve missing subtrahend tasks (eg 17-14 as 16, 15, 14 - answer 3). The child can choose the more efficient of count-down-from and count-down-to strategies.

Stage 5: Facile Number Sequence. The child uses a range of what we refer to as non-count-by-one strategies. These strategies involve procedures other than counting by ones but may also involve some counting by ones. Thus in additive and subtractive situations, the child use strategies such as compensation, using a known result, adding to ten, commutativity, subtraction as the inverse of addition, awareness of the "ten" in a teen number.

A1: Early Arithmetical Strategies. This aspect is based on extensive research by Steffe and colleagues (eg Steffe & Cobb, 1998; Steffe, 1992a). The model for this aspect appears in Table 2. Included in this aspect are:

(a) Initial counting strategies for establishing numerosity, or solving additive or subtractive problems, ie counting from one with items perceived, and counting from one without items perceived.

(b) Advanced strategies involving counting-by-ones, eg counting-up-from and counting-up-to — collectively referred to as counting-on, counting-down-from, and counting-down-to.

(c) Non-count-by-one strategies, ie cases where at least part of the strategy involves procedures that do not involve counting-by-ones, eg commuting an addition; using addition to work out a subtraction; using easier additions which are habituated, eg doubles; using known combinations to five or ten and knowledge of the tens structure of teens or of numbers from 20 onward.

A2: Base-Ten Arithmetical Strategies. Around the time they attain Stage 3 on the Model of Early Arithmetical Strategies children typically develop beginning knowledge of the tens and ones structure of the numeration system. Of course children can and should solve addition and subtraction tasks involving two-digit numbers long before they develop knowledge of the tens and ones structure. For children who have attained Stage 5, development of knowledge of the tens and ones structure becomes increasingly important. Table 3 outlines a progression of three levels in children's development of base-ten arithmetical strategies (Cobb & Wheatley, 1988).

Table 3
Model for Development of Base-Ten Arithmetical Strategies

Level 1: Initial Concept of Ten. The child does not see ten as a unit of any kind. The child focuses on the individual items that make up the ten. In addition or subtraction tasks involving tens, children count forward or backward by ones.

Level 2: Intermediate Concept of Ten. Ten is seen as a unit composed of ten ones. The child is dependent on re-presentations* of units of ten such as hidden ten-strips or open hands of ten fingers. The child can perform addition and subtraction tasks involving tens where these are presented with materials such as covered units of tens and ones. The child cannot solve addition and subtraction tasks involving tens and ones when presented as written number sentences.

Level 3: Facile Concept of Ten. The child can solve addition and subtraction tasks involving tens and ones without using materials or re-presentations of materials. The child can solve written number sentences involving tens and ones by adding or subtracting units of ten and ones.

* A re-presentation can be thought of as a mental replay (ie in reflection) of a prior experience that is distinct from and separated in time from the experience itself (See von Glasersfeld, 1987).

Part B

The three aspects in Part B are concerned with children's facileness on important specific aspects of children's early arithmetical knowledge: forward number word sequences (FNWSs); backward number word sequences (BNWSs); and numeral identification. There has been little systematic study of children's strategies relating to these aspects and thus little is known about the idiosyncratic nature and diversity of their strategies (Wright, 1998).

B1 & B2: FNWSs and BNWSs. From the point of view of fully understanding children's early arithmetical knowledge, it is useful to construe as distinct, the two aspects concerned with number word sequences, ie forward and backward. Models associated with these aspects are shown in Tables 4 and 5.

Table 4

Model for Forward Number Word Sequences (FNWSs)

Level 0: Emergent FNWS. The child cannot produce the FNWS from "one" to "ten".

Level 1: Initial FNWS up to "ten". The child can produce the FNWS from "one" to "ten". The child cannot produce the number word just after a given number word in the range "one" to "ten". Dropping back to "one" does not appear at this level. Children at Levels 1, 2 and 3 may be able to produce FNWSs beyond "ten".

Level 2: Intermediate FNWS up to "ten". The child can produce the FNWS from "one" to "ten". The child can produce the number word just after a given number word but drops back to "one" when doing so.

Level 3: Facile with FNWSs up to "ten". The child can produce the FNWS from "one" to "ten". The child can produce the number word just after a given number word in the range "one" to "ten" without dropping back. The child has difficulty producing the number word just after a given number word, for numbers beyond ten.

Level 4: Facile with FNWSs up to "thirty". The child can produce the FNWS from "one" to "thirty". The child can produce the number word just after a given number word in the range "one" to "thirty" without dropping back. Children at this level may be able to produce FNWSs beyond "thirty".

Level 5: Facile with FNWSs up to "one hundred". The child can produce FNWSs in the range "one" to "one hundred". The child can produce the number word just after a given number word in the range "one" to "one hundred" without dropping back. Children at this level may be able to produce FNWSs beyond "one hundred".

Table 5.

Model for Backward Number Word Sequences (BNWSs)

Level 0: Emergent BNWS. The child cannot produce the BNWS from "ten" to "one".

Level 1: Initial BNWS up to "ten". The child can produce the BNWS from "ten" to "one". The child cannot produce the number word just before a given number word. Dropping back to "one" does not appear at this level. Children at Levels 1, 2 and 3 may be able to produce BNWSs beyond "ten".

Level 2: Intermediate BNWS up to "ten". The child can produce the BNWS from "one" to "ten". The child can produce the number word just before a given number word but drops back to "one" when doing so.

Level 3: Facile with BNWSs up to "ten". The child can produce the BNWS from "one" to "ten". The child can produce the number word just before a given number word in the range "one" to "ten" without dropping back. The child has difficulty producing the number word just before a given number word, for numbers beyond ten.

Level 4: Facile with BNWSs up to "thirty". The child can produce the BNWS from "one" to "thirty". The child can produce the number word just before a given number word in the range "one" to "thirty" without dropping back. Children at this level may be able to produce BNWSs beyond "thirty".

Level 5: Facile with BNWSs up to "one hundred". The child can produce BNWSs in the range "one" to "one hundred". The child can produce the number word just before a given number word in the range "one" to "one hundred" without dropping back. Children at this level may be able to produce BNWSs beyond "one hundred".

B3: Numeral Identification. Learning to identify, recognise and write numerals may be regarded important in early literacy development but is also important in early arithmetical development. The term 'identify' is used here with precise meaning, ie to state the name of a displayed numeral. The complementary task of selecting a named numeral from a randomly arranged group of displayed numerals is referred to as 'recognising'. This corresponds with typical use of these terms in psychology and early literacy. Table 6 outlines a progression of four levels in children's development of numeral identification.

Table 6.
Model for the Development of Numeral Identification
Level 0: Emergent Numeral Identification.
Cannot identify some or all numerals in the range "1" to "10".
Level 1: Numerals to "10"
Can identify numerals in the range "1" to "10".
Level 2: Numerals to "20"
Can identify numerals in the range "1" to "20".
Level 3: Numerals to "100"
Can identify one and two digit numerals.
Level 4: Numerals to "1000"
Can identify one, two and three digit numerals.

Part C

C1: Combining and Partitioning. Counting strategies are an important aspect of children's early arithmetical knowledge. Nevertheless, children also develop knowledge of simple addition combinations, eg involving two addends in the range one to five, which does not rely on counting. Children learn to provide answers almost immediately to questions such as three plus three, using procedures that do not involve counting-by-ones. Arithmetical knowledge of this kind has been labelled "automatised" or "habituated". Recent research provides strong indications that teaching children to habituate simple addition facts through combining and partitioning of small numbers can significantly facilitate development of advanced arithmetical strategies, ie non-count-by-one strategies (eg Cobb, McClain, Whitenack, & Estes, 1995).

C2: Spatial Patterns and Subitising. Experiences with spatial patterns can be important in early number (eg Bobis, 1996). Children's strategies related to this aspect of early number arise in settings involving spatial configurations of various kinds, eg domino patterns, pairs patterns, Tens Frame, playing card patterns, regular plane figures and random arrays. Subitising is "the immediate, correct assignation of number words to small collections of perceptual items" (von Glasersfeld, 1982, p. 214). As explained by von Glasersfeld, when a young child for example, says "three" in response to a briefly displayed spatial array, the child may be doing no more than recognising and naming a spatial arrangement. One cannot assume that this child has a concept of "three". Nevertheless being able to name spatial arrays in this way is an important ability in early number (eg Bobis, 1996). No doubt at some point the child will see a correspondence between the name of the array, ie "three" and the last number word when they count the dots in the array, eg "one, two, three!". In early number there is a range of instructional settings for which spatial pattern or spatial arrangement seems to be a dominant feature. These include cards with random or irregular arrays, the Tens Frame, plane figures (triangles, squares etc.), and the various regular patterns (eg domino patterns for 1 to 6, pairs patterns for 1 to 10, and the patterns on playing cards).

C3: Temporal Sequences. Temporal sequences (von Glasersfeld, 1982, p. 201-204) involve events that occur sequentially in time, eg sequences of sounds and movements. Sequences of sounds may be rhythmical, arrhythmic or monotonic. Instructional settings in which the child's task is to count or copy a sequence of sounds or count a sequence of movements are considered likely to enhance early arithmetical knowledge. There has been little systematic study of children's strategies associated with temporal sequences. Experience has shown that, as a general rule, children are not as facile at counting temporal sequences as they are at counting items occurring in spatial sequence, eg rows of counters or dots. For example, in contrast to counting a row of counters, when counting a sequence of sounds the child has no control of the speed of perception of the individual items. Children will count slow monotonic sequences of sounds and sequences of movements similarly to the way they count a row of items, ie to coordinate a number word in sequence with their perception of each item. When counting rapid, rhythmical sequences of up to six sounds three strategies may be used: (a) to count the individual sounds from one as they occur; (b) to recognise the pattern in terms of its number of beats and thus answer without counting; and (c) to mentally replay or re-present the pattern, after its completion, and count the number of beats in re-presentation.

C4: Finger Patterns. Using one's fingers is very prominent in early arithmetic. There are variations in ways children move and make patterns on their fingers. For example, some will begin with five fingers raised, and lower fingers in turn while others begin with the five lowered and curled, and raise them in turn. Fingers are used in a range of ways and with varying levels of facility and sophistication. The current author has used the label "counting forward from one three times" for a prominent strategy involving perceptual counting, used to solve additive or subtractive tasks involving numbers up to 10. In the case of subtraction for example, a child might sequentially raise fingers while counting from one to the given minuend, then sequentially lower fingers while counting from one to the given subtrahend and then count the remaining fingers from one. In contrast, a strategy involving simultaneous raising of fingers to symbolise a number is more advanced. Another example of a more advanced use of fingers is the sequential raising of fingers to keep track of counts when using an advanced counting-by-one strategy (see earlier). This strategy, like the second last mentioned, also involves knowledge of a finger pattern and unlike the first mentioned, does not involve counting fingers sequentially to establish numerosity.

C5: Quinary-based strategies. These strategies involve using the number "five" as a base and arise in instructional settings involving the arrangement of items in fives, eg Arithmetic Rack (Gravemeijer, 1994, pp. 182-5) and Tens Frame, and also settings where use of fingers is prominent. Typically in these settings, the number "ten" is also a base. In early number there is a major potential advantage associated with "five" being used as an additional base along with "ten". Use of the base "five" has the potential to greatly reduce reliance on counting-by-ones. In other words, promoting the use of base-five in early number may well result in earlier development of facile arithmetical strategies, eg non-count-by-one strategies (eg Gravemeijer, 1994; Gravemeijer, Cobb, Bowers, & Whitenack, in press).

C6: Multiplication, division and fractions. As topics in early number these would reasonably be viewed as more advanced than addition and subtraction (eg Hunting, Davis & Pearn, 1997; Steffe, 1992b). Nevertheless, children in the early years (K-2) can and do reason multiplicatively (eg Mulligan & Mitchelmore, 1997) and in terms of fractions (eg Hunting & Sharpley, 1988). In its initial form the LFIN did not address this aspect but in its revised form it includes a section written by Mulligan (1997) focusing on young children's strategies in multiplication and division.

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