# MAKING SENSE OF PRIMARY MATHEMATICS

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*In this address I consider the research undertaken by me and by others which is concerned with how primary school children make sense of mathematics. The focus is on the strategies children use, the effect of processes and materials used by teachers, and the demand that these make on capacity to process information. The issues examined include the use of concrete representations, aspects of number, length and time measurement, and the transition from arithmetic to algebra. The results suggest that the strategies and materials that teachers choose often do more harm than good unless they are used thoughtfully and carefully. On the basis of the results I argue for mathematics to be taught in the most meaningful, straightforward, interesting, way possible.*

I am delighted to have been invited to present this paper. It has provided me with the stimulus to review research I have carried out in early mathematics learning and to attempt to make explicit the beliefs and assumptions underlying what I have done and concluded. Some of you might not be so delighted because it is minimally contentious and to a certain extent challenges conventional wisdom.

For more than 20 years I have been concerned with how and why children make sense of mathematics. I no longer teach students to teach mathematics, rather, my concern now is generally with how people learn and that perspective will no doubt shape what I have to say. During those twenty or more years I have seen many innovations and, recycling of the same, and heard confident pronouncements about materials and methods that would improve children's mathematical learning. In fact none of this is new. The debate about how best to teach mathematics and whether or not concrete materials help or hinder the process has a long history (Szendrei, 1996). I have been caught up in believing some of the ideas myself but I have always been sceptical and therefore I have constantly questioned the value of what I, and others, have been doing. This perspective might in a sense be destructive, if you believe in something strongly enough you will usually make it work even if the approach is not perfect, but is that really the most sensible behaviour? So a great deal of the research that I discuss here is of a questioning nature and some of it at the microlevel. It focuses on what children should be able to learn at particular ages and why, what they actually know, and how they make sense of mathematics as a result of, or despite, what is occurring in classrooms or mandated by curricula. I have come to believe that the most important ingredient in teaching anybody anything is to find out what they already know and to persuade them, through activity and discussion, to be interested in changing their conceptions for something more useful and effective. I am a moderate constructivist and probably a latter day phenomenographer. Basically I believe that we need to know what each child knows at any point in time in order to teach effectively. Then we should go about the process in the most meaningful, straightforward, interesting way possible.

There is not really a great deal of content in the primary mathematics curriculum. The real objective in those years, in my opinion, should be to have children understand the basic concepts. This is not achieved by chopping the content up into manageable and often meaningless segments and then trying to support performance with materials that might hinder rather than facilitate the process. Some good mathematics teaching occurs in many classrooms but in others problems are caused which permanently influence children's beliefs and conceptions about learning mathematics and the motivation to do so. Let me give you an example of how this can occur although such a situation would probably not arise in quite such a stark way today.

Many years ago my eldest son came home from school quite upset. When I asked him what was the matter he said that his 'tables had fallen out of his pocket.' 'What do you mean fallen out of your pocket I asked?' 'Well the teacher gave us these tables to learn on a piece of paper and I've lost them.' 'That's no problem, I said, you can use this box of matches to work them out.' Quite a while and much grumbling later he had found out that you could actually derive these things yourself. Before that he was being lead to believe that multiplication tables came from some external authority, that all you had to do was learn them by repetition and give them back, that you could not do anything about them yourself and that there was no meaning or interest in them.

Let me give you a few historical examples of some other problems I have seen. I taught young children and teachers in New Guinea over a number of years. When I first went there in the 60's someone had sold the idea of Cuisenaire rods as a way of teaching number to New Guinean children. It had been demonstrated elsewhere that children could find answers to problems well beyond what one would normally expect of them using the structure of those little coloured rods. It seems that no one really questioned then whether children related what they were doing to any knowledge of the number system as long as they could come up with the right answer. I did not know anything at that time about cognitive capacity or processing demand although I had a vague notion of Piaget's stages. However it seemed to me that using Cuisenaire rods to teach early number concepts was imposing a lot of unnecessary information on children who were learning in a third language and whose counting system did not go much beyond '20' in a base 5 system (that is using a couple of number names in combination to arrive at '*10*' which was the fingers of a hand). So I mostly used the blocks to improve their knowledge of colour names, comparative lengths, and their fine motor skills. I set about approaching numbers and the number system through counting and grouping objects using the base ten system of ordinal and cardinal numbers in English. When I came across these blocks in the Northern Territory in the 80's I suggested they be used for creative activities or paving sections of the floor. This does not mean that I think Cuisenaire rods cannot be useful in teaching more advanced concepts such as fractions but they need to be used in an informed and consistent manner.

The next wave of materials to sweep over us in New Guinea, and subsequently in other parts of Australia, were those devised by Dienes. He came to Papua New Guinea and persuaded the educational hierarchy that use of his Logic and arithmetical materials would teach children to think logically, understand the number system and even to learn algebra with ease. We were lead to believe that his Multibase blocks (which I think actually had their origin in Montessori's beads) were particularly appropriate because there was a number of counting systems in the children's first languages that were in bases other than ten. The work with sets and Logic blocks was great fun but did not seem to lead directly anywhere, and the Multibase blocks in base 10 were only useful if they were used very carefully and consistently, and deliberately related to place value and algorithmic processes As described later they need to be used in the same way with other children to be of any use. The blocks in other bases and the algebra materials were very difficult even for the teachers to make sense of and were quickly abandoned. Again it seemed to me that the path to understanding basic mathematics was being deliberately obfuscated by a clutter of peripheral concepts that increased the processing load, and in most cases caused complete confusion, unless children chose to bypass them all and make their own sense of things.

To a certain extent Dienes (cf. 1960) justified his materials and procedures on the basis of developmental theories such as those of Bruner and Piaget. This caused some of us to believe that if we understood Piaget's work more thoroughly we would be able to devise curriculum content and teaching procedures that would be more effective for most children.

Whilst the notion of stages has been most illuminating in generally understanding what students might be capable of learning at particular ages and has underpinned programs such as the Nuffield Project, it has also been restrictive. It limited, in particular, what people believed could be taught to young children and it did not take into account what individual children might make sense of, and hence the knowledge they might acquire, depending on their capacity to process information and social and cultural factors.

Then came the work of the Neo-Piagetians and the belief that processing capacity at various ages, and the demand of tasks, could be quantified and that these seemingly exact formulae could be applied to children's development and hence to curriculum content to devise more exact matches between them. These theories have been useful and have provided a basis for explaining why some of the material based approaches I described above have not always worked as well as they should. It has provided a theoretical framework for some of my research but it does not allow us to explain fully the idiosyncratic way in which most children do or do not make sense of their environment and do or do not acquire mathematical knowledge.

What emerges generally from the Neo-Piagetian work above is that working memory (or short term memory as it is alternatively or additionally described) has a very limited capacity. In an information processing model of memory this is where conscious thinking occurs. The limited capacity of working memory puts constraints on how much information can be considered at any one time without overloading that capacity. This is particularly the case with young children, where items of knowledge are separate rather than chunked. In some of the research I describe it is obvious that the strategies and materials that are used are likely to overload capacity unless teachers firstly ensure that knowledge of them becomes automatic through practice. If that occurs then some freeing up of working memory will occur and they can be used more effectively.

## USE OF CONCRETE REPRESENTATIONS

Teachers are generally persuaded to use concrete materials as representations of mathematical concepts because, according to Halford and Boulton-Lewis (1992), they believe, or have been lead to believe, that they reduce learning effort and serve as memory aids, increase flexibility of thinking, facilitate retrieval of information from memory, mediate transfer between tasks and situations, indirectly facilitate transition to higher levels of abstraction and can be used to predict unknown facts. There are however some disadvantages in their use. Mapping from a concrete representation to a concept imposes a processing load on working memory, particularly if the representation is unfamiliar, and this can interfere with understanding a concept. If a representation is poor it can generate incorrect information. If the teacher does not understand the materials well, uses them infrequently or uses them inappropriately, they also cause problems. If the climate in the classroom is one where they appear to be remedial devices for children with problems then they are not useful either because most children will not want to be seen to need them.

Most of the arithmetic concepts taught in the early years of school, such as addition and subtraction operations, require two way mappings if materials are used. That is, if concrete representations such as MAB (Multibase Arithmetic Blocks) are used they must be mapped into the concept of a quantity and the verbal name for it, and then into the symbolic place value representation of the number. The procedure is not usually as well ordered as this. The child is often expected to start with symbols alone or quantities expressed in word problems, to map the symbols into concrete representations, to perform the operation with the concrete materials and to map the resulting quantity back into symbols (written or verbal). We described (Halford & Boulton-Lewis, 1992) the mappings required for 324 minus 179 using MAB blocks. I will illustrate this process in the presentation to demonstrate why I believe that probably the most difficult approach to two or three digit subtraction, in terms of the processing load, is to use concrete materials, and perhaps counting, to support limited place value knowledge, whilst trying to understand and perform an algorithmic process. The child can end up with a huge confusing mess of blocks, not to mention other confusion, which is a positive deterrent to their use. Alternatively if a child has a good understanding of the basic addition operation, and of place value, then mental strategies should make less of a processing demand because neither the written algorithmic procedure, nor concrete representations of the numbers, need to be considered. These mental strategies, however, at some stage need to be related to written algorithms and calculator procedures.

## ASPECTS OF NUMBER LEARNING

Fortuitously I have recently had a paper published in the Journal of Mathematical Behavior (Boulton-Lewis, 1998) which summarises, a little after the event, research I have conducted over the last ten years or so into children's understanding of various aspects of number. If you are interested in following up the details of any of those studies you will find the references listed there. The data were derived from structured interviews and analysed keeping cognitive processing demand in mind. I have proposed that an explanation for some of the difficulties experienced with concrete representations, as well as some of the strategies that teachers and children use, is that insufficient account has been taken of the processing load they impose, over and above the processing load of the basic operation, that their use initially entails. I have also suggested that we do not sufficiently take into account children's existing understanding and their preferred ways of making sense of new material. I will also touch on some research in children's understanding of length measurement and time where it seems that children make sense of these measures themselves with reference to common tools (c.f. Szendrei, 1996) as opposed to educational materials. Then I will briefly describe research in prealgebra and early algebra learning from a similar perspective. The intent of most of all this research was to find how children interpreted symbols and operations, how they represented them with concrete materials and what strategies they used to solve problems.

### Place Value

We focused first on understanding of place value by children aged 6 to 8 years in the first three years of school (Boulton-Lewis & Halford, 1992). Amongst other results we found error patterns for reading and interpreting digits in numbers such as 12, 13, 16, and 18 in years 1 and 2 and these were not surprising because of the conflict between the ways in which they are spoken and written in English. Generally there was more success across all years in interpreting and representing numbers such as 20, 30, 40 and even 23 and 27 than the 'teens' numbers. When the number of items of information that must be taken into account to read, understand and represent a number such as 13, as compared with 30, is considered it appears that at least one extra processing step is required. The figure illustrating this can be found in Boulton-Lewis and Halford (1992) and will be discussed in the presentation.

Because of the known difficulties with 'teens' numbers, curriculum guidelines often recommend teaching decade and other numbers before introducing them. Whilst this might avoid problems with 'teens' it fragments the ordinal sequence and probably stops children coming to grips with the meaning of the place value system and the ways in which it is read. A more meaningful approach, we suggested, despite the processing load, is to teach the numbers in normal sequence and to encourage the children and teachers to really talk about these strange numbers and how you represent, read and write them. Nothing new I hear you thinking but does this really happen in classrooms?

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|---|---|
| **Pretest for 20** | |
| Interviewer (I) | 'What's this number?' |
| Child (C) | 'Oh, twenty, that's easy. Two tens.' Placed two MABs under 2. |
| I | 'How many tens do you have?' |
| C | 'Two tens and no ones.' |
| I | 'Why do we write the 2 here and the 0 there?' |
| C | 'Ooh because if you put that number there (0 before 2) it would make 2 |
| **Posttest for 20** | |
| C | 'This is going to be fun, I'll do all different things. Now ten' (One MAB ten). He then put out 6 counters, 2 sticks, and one MAB unit. |
| I | 'What do you have there?' |
| C | '20...1, 2, 3, 4, 5, 6, 7, 8, 9 (Counted all single materials), 9.' |
| I | 'Nine, so nine and ten make twenty? Are you certain?' |
| C | 'Yes.' |
| I | 'How many tens?' |
| C | 'One.' |
| I | 'How many ones?': 'Nine.' |
| I | 'What does "20" tell you?' |
| C | 'Twenty.' |

The most unexpected result with regard to place value was that in years 1 and 2 only half the children changed from using less to more efficient materials to represent tens in numbers. Some children changed from using more to less efficient representations. Such behaviour is illustrated by the following protocols for one child at the beginning and end of year 2.

So what was happening here? The teachers in years 1 and 2 did encourage children to use a mix of materials to represent numbers and that might have caused the change. As a result, by not making strong mappings between digits in the tens place of a number, the spoken number name, and the materials that most efficiently represent 'ten', knowledge of these materials would not become automatic. Hence this would increase the processing load of using such numbers in operations because the child would need to count or think of most of the number in units. There was no problem with such by year 3 but the result suggests that strategies and materials that teachers use with the best intentions can cause problems.

## Subtraction

Subtraction was the focus of our next series of studies These studies were conducted with children in years 1, 2 and 3 and subsequently with children in years 4, 5, and 6. We found, in a preliminary study (Boulton-Lewis, 1993a), a sequence of solution procedures similar to that found by Carpenter and Moser (1982, 1984) that is proceeding from material to verbal to mental strategies. By year 3 not many children chose to use written algorithms. They rarely used concrete materials and preferred to use mental strategies including recall of facts and place value explanations and were more successful when they did. This suggested that children did not understand the algorithmic procedures and saw no connection between them and their successful mental strategies. I suggested above that the most difficult way to perform written algorithms was with the use of unfamiliar concrete materials and these results support that proposition. The children were not sufficiently familiar with any concrete representation to know the relations between it and a quantity, numeration or operations. However children demonstrated knowledge of place value and the structure of numbers which they could have been assisted to relate to algorithmic procedures.

We also identified some teaching and curriculum procedures that were causing confusion. These included backward counting, the recommended sequence in the curriculum for teaching algorithms, and children's inability to 'read' symbols such as '-' and '13'.

The backward counting procedure with and without concrete materials increases the processing demand of a subtraction task. It does this because, in counting backwards to subtract, some of the information must be processed in parallel whereas when counting forward the information can be processed serially. It can also be confusing because in order to arrive at the correct solution an extra count must be made at the end of the sequence or else the count must not include the first number of the minuend. Some examples of the kind of confusion that occurred are as follows.

1. 8-3=6 (using counters) I counted backwards 3 places 8, 7, 6 (year 1)
2. 16-8=10 (with fingers and toes and no knowledge of place value) 15, 14, 13, 12, 11 (on right hand) 10 (on left hand), 10, I was right (year 2).

It was recommended in the curriculum guidelines at the time, in years 2 and 3 in Queensland, that 2 and 3 digit subtraction be taught initially without any decomposition. It was also recommended that children should be encouraged to use a range of concrete materials and counting strategies as required. By the end of year 3 it seemed that children had practiced so many subtraction items where no decomposition was required that they could see no use for concrete materials and did not really understand place value either. They often verbalised the numbers in the tens and hundreds columns as units and seemed to think of them as separate single digit operations. Below are two examples of children solving 31-6. The first shows only partial understanding of place value and inefficient use of materials. The second shows good mental strategies and no use of an algorithm.

1. The child read 31-6 correctly, wrote 31, said you can't do this, you can't take 6 from 1.' Asked if materials would help she put out 3 MAB tens and 1 unit, changed 2 tens for units, counted off 6, forgot the other ten and answered 15.
2. The child wrote the operation as a vertical algorithm, did not use it and said 'I took 6 from 30 and that is 4 and then I added 1 on for the 1 in 31 so that's 25.'

The preceding results might have been peculiar to the teaching in that particular school therefore a further study using similar tasks and procedures was undertaken with a sample of children in years 1, 2, and 3 in three schools. The results indicated that children across the three schools were choosing, where possible to recall or solve subtraction tasks orally/ mentally without materials. They did this both correctly and incorrectly. When they did use materials the most frequent strategy was forward counting. In a few cases where they chose to use written algorithms they usually did not use materials and mostly used 'buggy procedures' when decomposition was required. Their use of mental strategies was more successful and apparently not related to use of written algorithms. A MANOVA was undertaken for the most frequently used strategies early and late in each year. The details of these can be found in Boulton-Lewis (1993b), however the general developmental sequence was again from material (using objects and counting) to verbal (using only oral counting) to mental (using known number facts and place value) as found in other places by Carpenter and Moser (1982, 1984) and De Corte and Verschaffel (1987). However we found that materials were used consistently throughout the three years, that children tried to recall and use number facts as early as possible, and that by year 3 the mental explanations were mostly based on knowledge of place value and decomposition of tens and hundreds, and were used in preference to written algorithms. In summary the children did not use written algorithms with any confidence or success. Neither were they making connections between these, their mental strategies, and the materials that the teachers wanted them to use. The results were similar to those that Hart (1989) found in the United Kingdom. It is suggested that this is in part because the demand imposed by concrete materials was not considered and addressed. We also believe that teaching subtraction operations for numbers with 2 and 3 digits without the need for decomposition, until they are performed automatically without understanding, obviates the need to consider place value and to use materials that would, if used properly, facilitate understanding.

A further study was conducted for subtraction in years 4, 5, and 6 (Boulton-Lewis, Wilss & Mutch, 1996a). The general frequency of strategy use, in decreasing order, was from material to mental to written strategies in all years. This was similar to the sequence found for years 1 to 3 except that there was now a move to using written procedures. Materials were used more at the beginning of each year and this could have been due to the perceived difficulty of increasingly larger multidigit subtraction items, increasing familiarity with the materials and increasing capacity to process information. This sequence would have been influenced by teachers' behaviour and expectations however the tendency to adopt mental strategies continued despite, or perhaps because of, what teachers do. There was a decline in the use of materials in year 6 and this was probably due to better overall understanding of the subtraction process and recognition that the task can be performed mentally or with a written algorithm. On the basis of the results of this study and earlier work it was suggested that if algorithms were treated minimally in the early years and if much greater use was made of mental strategies, with and without concrete materials, then a range of algorithms could be explored later with greater understanding, and as an enjoyable way of relating to and confirming mental procedures. It has been suggested in the literature that children should be introduced to a range of algorithmic procedures, presumably, so that they can develop a deeper understanding of the process. For example working from left to right to solve subtraction could be tried. Evans (1990) observed second and third grade children working on two digit subtraction algorithms and noted that two thirds began working in the tens column. Baroody (1987) advocated this procedure for carrying out mental computations and Fuson (1990) suggested that subtraction taught from either direction is viable. Decomposition is typically taught to American [and Australian] children as a right to left process. We suggested that interest in the use of algorithms is probably killed when children must rely on the support of concrete materials and are hindered in understanding the process by the demand that makes initially on processing capacity. Children in this study were also using calculators to link mental and written procedures but their use declined over the three years as facts and procedures became better known.

## Addition

A study of addition (Boulton-Lewis & Tait, 1994) was undertaken with the same sample of year 1, 2 and 3 children as those for the subtraction research described in Boulton-Lewis (1993b). The results showed similarly that children generally chose to use material strategies and count until year 2, to use the verbal strategies of recall of facts and derived facts until year 2, and then to use place value explanations (mental ) or joining of sets of objects and counting (material) as frequently at least as written algorithms in year 3, with or without concrete materials despite the fact that they were being 'taught' to use the written algorithm. When errors were analysed it was apparent that they occurred when children did not understand place value, chose to use written algorithms without materials or without understanding, used place value explanations without materials or just used large amounts of discrete materials. The results indicated, just as they did for subtraction, that children used verbal and mental strategies in preference to formal algorithms and did not want to use materials unless they could not perform the task in any other way. The results are encouraging in that children are making sense of the operations using their own strategies. They apparently reduce the processing load by using no materials or materials only. The results are not so good if the objective as proposed in the curriculum guidelines is to have children relate their prior knowledge of the addition operation to the recommended formal written procedures.

## LENGTH AND TIME MEASUREMENT

### Length

We (Boulton-Lewis, Wilss & Mutch, 1996b) also interviewed children in years 1, 2 and 3 to determine their use of strategies and devices for measuring length. They were given a range of materials (both common tools and educational materials as described by Szendrei, 1996) and three tasks that required respectively, comparison or direct measurement, direct measurement, and indirect measurement or consideration of size and number of match sticks fixed to cards. They were asked to explain their procedures as they worked on tasks. Our results showed that the most preferred strategy was to use a standard measuring device (common tool) such as a 30 cm ruler initially in an arbitrary way in all years. It was the most frequently chosen device in years 1 and 3 and the second most frequently chosen in year 2 despite the fact that teachers were encouraging children to use arbitrary devices such as hand spans, paper clips, or other discrete objects as recommended in the curriculum. Visual perception, sometimes with counting, was used quite heavily in year 1 and less so in subsequent years.

On the basis of theories of cognitive development, and what we knew about learning in and out of school, we questioned the recommendations in curriculum documents about teaching length measuring to young children. It is suggested that children should initially undertake a range of measurement activities with arbitrary units so that they will see the need for standard units. An analysis of the processing load of such tasks suggests that the procedure would defeat the purpose of what it is intended to achieve. In order to realise that arbitrary units are not reliable children must reconcile the varying lengths and numbers of arbitrary units and reason transitively. Alternatively to use a standard device such as a 30 cm ruler to make a pair by pair direct comparison of lengths of objects is a less demanding task. In addition in earlier research I found (Boulton-Lewis, 1987) that children could successfully measure using a ruler before they could devise a strategy using arbitrary units and before they could reason transitively. Using a standard device also has the advantage that it is a common tool and children see adults using such measures, therefore it appears to be part of meaningful real world activity. Our results confirmed these predictions and showed that children apparently prefer to use a standard measuring device even if they do not understand it fully or use it accurately. We suggested that children should be encouraged to measure directly and indirectly with standard and arbitrary measures from the first years of school. To avoid the kinds of errors of usage associated with standard measures described in the literature such as incorrect alignment of the ruler and the object to be measured, focussing on the numbers on the ruler rather than the standard unit of measurement, starting at a point other than zero, and leaving gaps between rulers, teachers should discuss explicitly with children how to use standard measures. It seems that these results also provide a strong case for the more complex concept of the construction of, and need for, a standard measuring device to be left until children are about 8 or 9 years of age. It also suggests that measuring activities would provide more interest for children if they were allowed to use standard measures in more or less real world situations.

### Time

Children from years 1-6 were tested for their ability to read and record analogue and digital times (Boulton-Lewis, Wilss & Mutch, 1997). The children in year 6 were also asked to describe their strategies. Based on a review of the literature and cognitive theories a sequence for acquisition of time concepts was proposed. This was hour, half hour, quarter hour, five minute and minute times. It was proposed that times after the half hour would be more difficult and digital times would be learned sooner. This sequence was confirmed for years 1-3 but irregularities occurred in years 4-6 where it was obvious that there was greater individual difference in ability to read and record analogue times. Although formal instruction for reading time is not recommended until year 2 there was an improvement in reading time from years 1-3 and most children could read hour times in year 1. Minute times proved to be the most difficult although it was recommended in the curriculum that these be taught in year 2 along with hour times. The majority of children achieved success earlier for digital reading than analogue reading. This did not necessarily mean that they understood what they were reading. By year 4 it was apparent that understanding of both representations of time was developing as children were starting to use strategies such as calculating based on 60 minutes and referring to the analogue clock when explaining the reading and recording of digital times. In years 4-6 a major error was in reading the hour hand for analogue times and this suggests the need for explicit instruction. The results show that children know more about time at an earlier age than teachers and curricula give them credit for, that strategies could be investigated to determine those which produce greater understanding and that teachers should not assume that all children understand all aspects of reading and recording time. The results made eminent sense in terms of the difficulties involved, the information processing analysis and the kind of time representation that children see being used around them.

## TRANSITION FROM ARITHMETIC TO ALGEBRA

We have carried on the research in the use of materials and strategies and applied it to prealgebra and early algebra learning in the last year of primary school and the first two years of secondary school. Although this exceeds the brief that I have, to talk about primary school mathematics, I want to mention briefly some of our results. Some of the problems that we found in understanding prealgebra and algebra concepts relate to the use of concrete representations and poor understanding of basic arithmetic operations originating earlier in the primary years. When we undertook a pilot study of the use of concrete materials to teach algebra in year 8 (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss & Mutch, 1997a) we found that students did not use the procedures taught to them for concrete representations, no students used materials voluntarily, and the preferred mode of thinking to solve unknowns in equations, on both pre and posttests, was to use arithmetic strategies mentally both correctly and incorrectly. We explained the lack of use of materials primarily on the basis of the processing load associated with them. In the school where we undertook this study we found that a range of representations and strategies had been recommended to teachers to facilitate the teaching of algebra. We believe that some of them would be useful and that some are questionable (for example the use of different coloured counters to represent directed numbers and variables in equations). MacGregor and Stacey (1995) asserted that ‘many recommendations in the pedagogical literature [for teaching algebra] ... have no supporting research background’ (p82). They found for example that the use of patterns in primary and later years as a foundation for algebra did not lead to the understanding that might be expected. We believe that as with all use of representations the relative cognitive demands must be considered. As with other materials they need to be used in a consistent explicit way in which the concepts related to the materials are abstracted and discussed. We saw them being used in a random and indiscriminate fashion on the implicit assumption that you could use a representation once to represent an aspect of algebra, use something quite different the next time, and expect the children to abstract and integrate the algebraic concepts themselves. We discuss a range of these materials and strategies in (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss & Mutch, 1997b). I will consider one of these here, that is, the use of tables of variables to teach functional relationships between two variables. It has been argued that introducing algebra in this way facilitates its use as a language for expressing relationships. However it requires students to understand the concept of variable, and the syntax of algebra [or arithmetic at least], to take into account the two operations at once and the relationship between them. and in some cases to relate these to concrete representations. It is obviously a more demanding task than the use of a linear equation with one unknown and it is no wonder that MacGregor and Stacey found that many students had difficulty generating algebraic rules from patterns and tables. It is even less surprising when one considers the faulty knowledge of necessary arithmetic operations that we found with year 7 students.

Subsequent to the pilot study we undertook a three year study of the transition from arithmetic to algebra. Only the results for 51 year 7 students in three primary schools are considered here (Boulton-Lewis et al., 1997b). This part of the study provided information about knowledge of operations and operational laws, equals, pronumerals (unknowns), variables and solution strategies for linear equations. It was assumed that the students would have been taught about operations and equals as part of the curriculum in arithmetic but that any knowledge of variables and linear equations would only have been derived intuitively from their knowledge of arithmetic. We found that most of the students had sufficient understanding of the basic binary operations, or sequences of them for subtraction, multiplication and addition, in that order to be able to use them as a basis for algebra. Some students did not have sufficient understanding of division. Only two thirds of the group had sufficient understanding of the inverses of multiplication/division and addition/ subtraction. About 20-25% of the sample had sufficient understanding of the correct order of arithmetical operations to allow them to apply this satisfactorily to learning linear equations. With regard to the 'equals' sign in a completed equation only half the students knew that it meant both sides were equal. For an incomplete equation, with a series of operations, all the students believed it meant find the answer. This means that in subsequent algebra learning most would want to find the answer after the sign and at least half would need to learn the concept of equivalence. More than half the sample could solve an equation with □ as an unknown number or knew intuitively that it was like an $x$ or $y$ despite no explicit instruction in variables. The majority of students knew what $x$ was in an equation. When asked to use concrete materials most of the students 'illustrated' arithmetical solutions.

We proposed that the most accessible route to algebra could be through solving simple linear equations arithmetically to find unknowns. We know that this runs counter to most current thinking but it fits with Sfard's operational perspective of algebra (Sfard & Linchevski, 1994) and has the advantage that the transition can be made without the need for concrete representations except in the early stages of learning about numbers and arithmetic and perhaps to illustrate the concept of variable. It is also in keeping with the developmental sequence proposed by Halford and Boulton-Lewis (1992). We have explored a two path model for the transition in other papers (eg. Pillay, Wilss & Boulton-Lewis, 1998).

## CONCLUSION

In conclusion I have a few simple suggestions for making mathematics more meaningful for children in primary school and beyond. As I stated at the beginning it seems to me that there is not a great deal of essential content to master and that we should aim for real understanding of the basic material rather than coverage and activities that often do not seem to make any sense. I believe that some curriculum and teaching decisions are giving children the wrong beliefs and conceptions about learning mathematics. We also seem, except in the best classrooms, to be depriving children of motivation by suggesting that many aspects of mathematics are derived from some external authority, are not directly accessible by children themselves and have little to do with any worthwhile activity in the real world. We could remove the artificial constraints from curricula. That is we could set general targets for children to meet but make it clear that they should be encouraged to go beyond these and not be forced to perform meaningless tasks such as 2 and 3 digit subtraction without decomposition. We could also give mathematics real meaning by setting up situations such as those described by Resnick, Bill and Lesgold (1992). They argued that human mental functioning must be understood as fundamentally situation-specific and context-dependent. They assessed the early mathematical knowledge that children brought to school and designed a programme that drew on that knowledge, developed children's trust in their own knowledge, used correct formal notation from the start to link that knowledge to the formal language of mathematics, and introduced key mathematical concepts as quickly as possible (including multi digit subtraction in the first year). At the same time they allowed children to use manipulatives or expanded notation as they chose, encouraged everyday problem solving, and caused children to talk about mathematics rather than just do it. The teaching involved a great deal of discussion guided by the teacher in what they described as a 'kind of cognitive apprenticeship approach'. This obviously assisted the children to use a range of strategies effectively because the year 1 children in their programme, who were not restricted to a limited range of numbers, performed significantly better on standardised tests than a similar group of children who were taught in a more conventional way. So I am suggesting that we rethink what we are doing, remove complicating strategies and unnecessary materials and teach so that all aspects of mathematics in the early years are seen as part of a meaningful system. We should also keep in mind the demand that various operations and strategies will make on children's capacity to process information but at the same time recognise that motivation, provided by ownership of real problems and acknowledgment of children's existing mathematical knowledge, might allow children to explore and make better sense of mathematics by themselves with a little help.

I can still remember the delight evidenced by another of my sons, aged six, when he 'borrowed' his father's measuring tape, climbed without permission to the top of a two metre cement wall and crawled along the top measuring lengths in centicubes (he was using them at school) and used numbers way beyond those prescribed in the curriculum. This was an exciting and meaningful activity for him in more ways than one. I was pleased he survived the experience and learned from it. I am not suggesting that we encourage such dangerous activities but I would like to see children believe they can take some control of what they learn in primary mathematics.

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