ASSESSING DIAGRAM QUALITY: MAKING A DIFFERENCE TO REPRESENTATION

Carmel M Diezmann
Centre for Mathematics and Science Education
Queensland University of Technology
<c.diezmann@qut.edu.au>

The effectiveness of a diagram in problem solving is dependent on its utility as a cognitive tool. In order to develop students' ability to use diagrams as cognitive tools, teachers need to assess the quality of students' diagrams and provide them with the necessary support. However assessing the quality of diagrams is problematic. This paper discusses how theoretical prototypes and exemplars of level of performance provide a practical and effective avenue for assessing the quality of students' diagrams.

Mathematics educators strongly advocate the use of the strategy *draw a diagram* for mathematical problem solving (e.g., National Council of Teachers of Mathematics [NCTM], 1989). This perspective is strongly grounded in the belief that generating a diagram facilitates the conceptualisation of the problem structure (van Essen & Hamaker, 1990). Although the use of a diagram as a tool of mathematics can empower primary students to deal with novelty (NCTM, 1989), effective problem solving depends on the quality of students' diagrams (Yancey, Thompson, & Yancey, 1989). Thus, the generation of high quality diagrams should be a goal in any instructional programme on diagram use. The salient question that emerges, which is the focus of this paper, is how to assess the quality of students' diagrams.

THE USE OF THE DIAGRAM IN PROBLEM SOLVING

The advantages of generating a diagram are related its utility as a cognitive tool (e.g.; Larkin & Simon, 1987; van Essen & Hamaker, 1990). For example, diagrams act as an external sketch pad where interconnected pieces of information can be chunked together in a holistic manner (van Essen & Hamaker, 1990). Thus, implicit information within a problem may become explicit to the solver on a diagram (Larkin & Simon, 1987). However not all diagrams have the potential to be cognitive tools. For example, some students generate diagrams that focus on the surface (literal) features of the problem at the expense of representing the problem structure (e.g., Dufoir-Janvier, Bednarz & Belanger, 1987).

Problem representation can be enhanced by knowledge of general purpose diagrams, namely, *matrices, networks,* and *hierarchies*, and a range of diagrams that exhibit *part-whole* characteristics (Diezmann, 1999; Novick, Hurley & Francis, in press). These diagrams assume an important role in mathematics because they provide representational frameworks that are appropriate for a range of problem structures. For example, because of its particular spatial characteristics, a matrix can be used to represent the problem structure in combinatorial tasks (English, in press) or in deductive reasoning tasks (Novick, in press). Networks and hierarchies are also spatially-oriented diagrams that have unique visual characteristics. A network is a path-like representation (e.g., a train line map), whereas a hierarchy is a tree-like representation (e.g., family tree) (Novick, in press). Knowing the conditions of applicability for each of these representations is advantageous in selecting an appropriate diagram type, which is the first step in generating a successful diagram (Diezmann, 1999). Part-whole diagrams, unlike matrices, networks and hierarchies, have no unique visual characteristics. For example, both a diagram showing a "pie" with a quarter of the pie missing and a Venn diagram could be categorised as part-whole diagrams. There can be considerable variation between correct representations of a particular "part-whole" problem however. For example, in representing the number of tennis players in a class, there are various ways to draw the tennis players and the whole class, and to represent the relationship between them. Due to the range of diagram types and the variation possible within some diagram types, assessing the quality of diagrams can be problematic. However to support the development of students' use of the diagram as a cognitive tool, teachers need to be able to assess the quality of students' diagrams.

ASSESSING THE QUALITY OF A DIAGRAM

The quality of a diagram can be determined by the correctness of the diagram as a representation for the problem (Lindvall, Tamburino, & Robinson, 1982; van Essen & Hamaker, 1990). Correctness involves selecting an appropriate diagram type and generating a diagram that accurately represents the problem structure (Novick, in press). The appropriateness of the diagram can by determined from the conditions of applicability of particular diagrams (Novick, in press). For example, a hierarchy is an appropriate diagram for representing a knock-out tennis competition. However determining the accuracy of the diagram is more complex, due to the surface variation that may occur between diagrams, as discussed previously.

Lindvall, Tamburino, and Robinson (1982) proposed that the focus in assessing the accuracy of students' diagrams should be on the representation of the quantitative and qualitative components of the problem structure. Thus, for each problem situation, there are essential components of the problem that should be accurately represented on a diagram. These components constitute a theoretical prototype for the problem representation. Prototypes are particularly useful for ascertaining expertise when the degree of similarity between the exemplars (i.e., diagrams) may be low (Sternberg & Horvarth, 1995). The congruence between the diagram and the prototype indicates the students' level of performance in diagram generation and additionally, identifies aspects of diagram generation that need further development.

The purpose of this paper is to explore how the quality of diagrams can be assessed using theoretical prototypes, and specifically, how prototypes can be used to identify different levels of performance. Pragmatically, it is useful for teachers to have a set of performance benchmarks for assessment purposes (Maher & Martino, 1992). This "benchmarking" process will be illustrated using one problem in which a spatially-oriented diagram (e.g., matrix) is appropriate and another in which a conceptually-oriented diagram (i.e., part-whole diagrams) is appropriate.

METHOD

The establishment of performance levels for students' generation of general purpose diagrams was a component of a case study (Yin, 1994), which evaluated the effectiveness of instruction on various aspects of diagram use in novel problem solving (Diezmann, 1999). In the case study, it was hypothesised that there would be an improvement in students' generation of diagrams after instruction. Hence, it was necessary to develop a means of ascertaining the quality of students' diagrams on the pre- and post-instruction isomorphic tasks in order to compare students' performance and test the hypothesis.

The participants in the case study were 12 Year 5 students with a mean age of 10 years 3 months from a moderately sized school in a lower socio-economic suburb in Brisbane, Australia. They represented a cross section of students, who were high and low performers in problem solving, and had high and low preferences for a visual method of solution. The instruction consisted of twelve half-hour whole class lessons and addressed the generation and use of the four general purpose diagrams on novel problem solving tasks. Tasks comprising isomorphic sets of five novel problems were presented to each participant during 30 minute interviews conducted before and after instruction. The interviewer was known to the subjects through prior classroom involvement. The interviews were videotaped and subsequently transcribed. As the participants were not specifically instructed to use a diagram, those participants who did not spontaneously use a diagram were given further opportunities to generate a diagram. However the data used in the establishment of levels of performance for the accuracy of diagrams, is restricted to the diagrams that were spontaneously generated by the students.

The students' diagrams from the interview tasks were classified according to the degree of congruence between their diagram and the theoretical prototype on the following basis. When no diagram was produced Level 0 was assigned. Students' diagrams that could plausibly be considered to be of the appropriate type but had no structurally accurate components were assigned Level 1. The diagram was assigned Level 2 when at least one but not all of the structural components was represented accurately. Diagrams in which all structural components were accurately represented were assigned Level 3.

RESULTS AND DISCUSSION

Two examples of the diagram classification process are presented for discussion. The first example involved the use of a spatially-oriented diagram (a matrix) and the second example involved a conceptually-oriented diagram (a part-whole diagram). Both the problems that are discussed were post-instruction tasks because on these tasks the students generated diagrams for the full range of levels of accuracy.

The Sports task is a deductive problem with a factorial structure comprising two sets (see Figure 1). Tasks with this structure can be represented using a matrix (Novick, in press). The theoretical prototype for this task consisted of *two distinct sets,* which are drawn as a two-dimensional representation consisting of *rows and columns*. One of the sets should be represented on a row and the other on a column. Although the location of each of the sets, either on a row or a column, is irrelevant, each member of the set needs to be represented in the same location as the other members of that set. For example, on the *Sports* task the four people should be represented either on a row or a column (see Figure 2).

Figure 1
The Sports task.

Sports: *Four friends like different sports. One likes tennis, one likes swimming, one likes running and one likes gym. Each person only likes one sport. Use the clues to help you find out which sport each friend likes. Sally and Rick met when one of them won a swimming race. Tara and Greg met when one of them was exercising at the gym. Sally is not a swimmer or a runner. Greg is a friend of the gymnast's brother.*

Students' performance on this task confirmed that four levels of accuracy in the generation of a matrix could be identified. On Figure 2, these levels are described and students' diagrams are presented and annotated.

The assignment of levels for the matrix task was straightforward because a gradual development in the accuracy of the matrix was evident from Levels 1 to 3, which is the optimal level. The only variation in the representation of information on a matrix is in the positioning of each set of information. For example, in each of the diagrams on Figure 2, the students positioned the sports in the columns, however, this set could also have been positioned on the rows. A change in position of the sets would not affect the assignment of levels because position is not a critical aspect in the representation of sets on a matrix. Hence, despite differences in accuracy, the students' "matrices" were visually similar.

In contrast to the *Sports* task, *The Park* problem can be represented with a part-whole diagram (see Figure 3). A theoretical prototype for *The Park* comprised the *representation of parts of the total* (i.e., legs), and the *representation of the total* (i.e., total number of legs). As with Task 1, the students' diagrams varied substantially in their congruence with the prototype (see Figure 4). However, the assignment of levels on the part-whole tasks was more complex than on the matrix task (Task 1). Although there were the same number of levels of accuracy on both tasks, there was considerably more surface variation in the part-whole diagrams compared to the matrices. For example, on Figure 4, students represented "legs" as lines, as part of an animal, and as dots. Additionally, there was variation in the grouping of these legs to make a person or a dog. Groupings were represented by circling, by attachment as part of a person or dog, and by the positioning of the legs (dots) near lines. Furthermore, there was ambiguity in the use of graphic components. Whereas, Ian and Gemma used a detached line to represent a "leg", Damien used a detached line to represent either a person or a dog. Whether Damien's line was a person or a dog depended on the number of dots near the line. Two dots near a line represented a person and four dots near a line represented a dog.

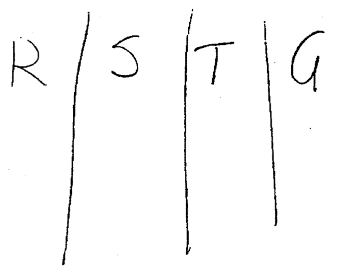
Figure 2
Four levels of performance for student-generated matrices.

Level 0

No diagram was drawn

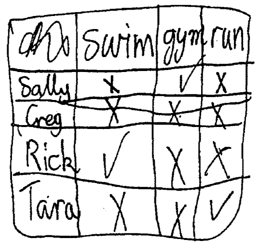
Level 1

Two distinct groups are not represented, however there is some similarity between the diagram and a matrix. *Lisa's explanation of this diagram revealed that she had represented the different sports using their initial letters but had not represented the people.*



Level 2

Two distinct groups are represented but there is an error in the representation. *Gemma has correctly represented the people on the matrix but has omitted tennis from the set of sports*



Level 3

Two distinct groups are represented correctly. *Helen has correctly represented the sets of people and sports, and the relationship between each set.*

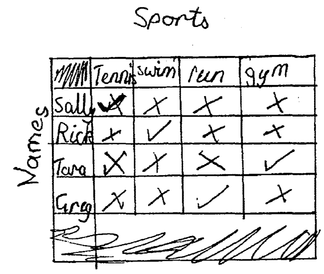


Figure 3
The Park task.

***The Park:** Jane saw some people walking their dogs in the park. She counted all the legs and found that there were 48 legs altogether. How many people and how many dogs?*

Visual differences in the surface details of students' part-whole diagrams complicate the assessment of the quality of the diagram. For example, although Ian's and Gemma's diagrams were visually similar, Gemma's diagram was of higher quality than Ian's because she accurately represented the total number of legs whereas he did not. Furthermore, although Candice's and Gemma's diagrams were visually distinct they were both assigned the same level albeit for different reasons. Thus, in determining the "accuracy" of part-whole diagrams attention to the theoretical prototype is required.

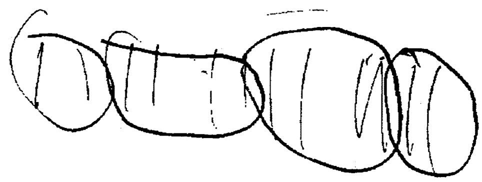
Figure 4
Four levels of performance for student-generated part-whole diagrams.

Level 0

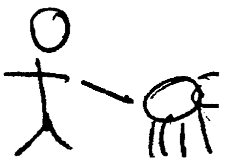
No diagram was drawn.

Level 1

A diagram was generated but neither the sets (the parts) nor the total (the whole) were correctly represented. *Ian's diagram was intended to represent the groupings of legs. However neither the number of legs nor the groupings was correct.*

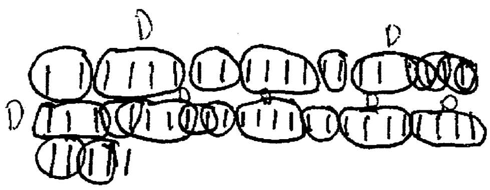


Both of the sets were correctly represented but the total was either omitted or incorrect. *Candice represented one of each set of legs correctly, however the total was not represented*



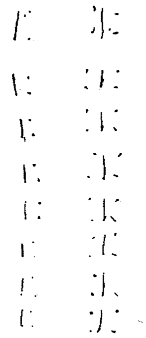
Level 2B

The total was correctly represented but at least one of the sets was incorrectly represented. *Gemma represented the correct number of legs however the grouping of legs was incorrect because there was a leg that was not included in the groups.*



Level 3

Both the sets and the total were correctly represented. *Damien correctly represented the sets of legs and the total number of legs. A line with four dots represented a dog and a line with two dots represented a person.*



CONCLUSION

The development of theoretical prototypes and levels of performance has the potential to make a difference to teachers' ability to assess the quality of diagrams and students' use of the diagram as a cognitive tool in the classroom. Knowing a student's level in diagram generation for particular types of problems and across the range of general purpose diagrams enables the teacher to pinpoint a student's difficulties, to provide effective intervention, and to monitor the development of diagrams as a cognitive tool in mathematics.

The theoretical prototypes for these tasks provided a useful guide for establishing the quality of students' diagrams. However the need for the prototype varied. On the matrix task, the theoretical prototype provided the guidance needed to establish a series of levels of performance. However after establishing these levels, the theoretical prototype is essentially redundant because subsequent diagrams can be classified visually without reference to the prototype.

On the part-whole task, the prototype assumes an ongoing role. Due to surface variation that occurs with part-whole diagrams, the prototype provides a needed reference point during the assessment process. For example, the prototype provides the justification for assigning Candice's and Gemma's diagrams the same level despite their substantive differences (see Figure 4). The prototype also indicates the particular support they need to develop their ability to generate effective diagrams. Whereas Candice needs to represent the total in her diagram, Gemma needs to represent the parts of the total accurately. Knowing this information is advantageous for teachers because specific components can be addressed during instruction and students' progress in representing these components accurately can be monitored.

Differences in the ease of assessing matrices and part-whole diagrams can be explained by the orientation of the diagram. Because matrices have unique visual characteristics, they are easier to assess visually, than part-whole diagrams that lack unique visual characteristics. The relative ease of assessing hierarchies and networks, which are also spatially-oriented diagrams (Diezmann, 1999) suggests that the various spatially-oriented diagrams are relatively easier to assess visually than conceptually-oriented diagrams.

Formative assessment serves a crucial role in the learning process. If students are to use diagrams as a cognitive tool, teachers need ways to determine what support is required by learners to achieve this goal. By assessing diagram quality and providing the necessary support, teachers can make a difference to problem representation (Diezmann, 1999), which is a crucial aspect of successful problem solving (Nickerson, 1994). Developing prototypes of essential problem components and identifying levels of performance in diagram generation provides teachers with a practical avenue for understanding and addressing some of the issues inherent in assessing the quality of students' diagrams and improving the effectiveness of diagrams as cognitive tools.

REFERENCES

Diezmann, C. M. (1999). *The effect of instruction on children's use of diagrams in novel problem solving*. Unpublished PhD thesis, Queensland University of Technology.

Dufoir-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Representation in the teaching and learning of mathematics*. (pp.109-122). New Jersey: Lawrence Erlbaum.

English, L. D. (in press). Assessing for structural understanding in children's combinatorial problem solving. *Focus on Learning Problems in Mathematics*.

Larkin, J. H., & Simon, H. A. (1987). Why a diagram is (sometimes) worth ten thousand words. *Cognitive Science, 11*, 65-99.

Lindvall, C. M., Tamburino, J. L., & Robinson, L. (1982, March). *An exploratory study of the effect of teaching primary grade children to use specific problem solving strategies in solving simple arithmetic story problems*. Paper presented at the Annual Meeting of the American Educational Research Association, New York. (ERIC Document Reproduction Service No. ED 216 904)

Maher, C., & Martino, A. (1992). Teachers building on students' thinking. *Arithmetic Teacher, 39*(7), 33-37.

National Council of Teachers of Mathematics. (1989). *Standards*. Reston, VA: The National Council of Teachers of Mathematics.

Nickerson, R. S. (1994). The teaching of thinking and problem solving. In R. J. Sternberg (Ed.), *Thinking and problem solving* (pp. 409-449). San Diego, CA: Academic Press.

Novick, L. R., & Hurley, S. M., & Francis, M. (in press). Evidence for abstract, schematic knowledge of three spatial diagram representations. *Memory and Cognition*.

Sternberg, R. J. & Horvarth, J. A. (1995). A prototype view of expert teaching. *Educational Researcher, 24*(6), 9-17.

van Essen, G., & Hamaker, C. (1990). Using self-generated drawings to solve arithmetic word problems. *Journal of Educational Research, 83*(6), 301-312.

Yancey, A. V., Thompson, C. S., & Yancey, J. S. (1989). Children must learn to draw diagrams. *Arithmetic Teacher, 36*(7), 15-23.

Yin, R. K. (1994). *Case study research: Design and methods* (2nd ed.). Thousand Oaks, CA: Sage.