MENTAL ADDITION AND SUBTRACTION STRATEGIES: TWO CASE STUDIES

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This paper tracks two children's mental strategies over a 5 year period (Years 2 to 6). Although the children were students in traditional classrooms, where teacher-taught algorithms may have conflicted with the children's spontaneous strategies, they continued to develop their own efficient strategies throughout the period of the longitudinal study. However, by Year 6, both children were also employing taught pen and paper algorithms which were less effective for mental calculations. Finally, some implications for teaching are discussed.

Ample evidence exists in the literature to suggest that children have difficulties learning computational algorithmic procedures by traditional transmission teaching methods (e.g., Olivier, Murray, & Human, 1990), and understanding what is happening when they do learn the procedures by the traditional methods (e.g., Resnick & Omanson, 1987). Contrary to this conclusion, there is evidence from research in reform classrooms that active involvement in mathematics learning enables children to understand computation, particularly where formal mathematical knowledge is built on informal knowledge (Carraher, Carraher, & Schliemann, 1987; Carroll, 1997; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, & Fennema, 1997; Kamii, Lewis, & Livingston, 1993; Thompson, 1994).

Carroll (1997) and Kamii, Lewis, and Livingston (1993) documented the mental and written computational procedures invented by children who are active in their learning. They showed that children can produce a wide variety of efficient strategies that exhibit sound number understanding even though there was little direct teaching of algorithms. They also found that the active development of knowledge encouraged children to participate in the construction of problems and the explanation of solution strategies.

In Queensland, there are few reform classrooms and the traditional pen and paper algorithms are still taught out of context and in situations where children have little or no input into constructing problems and explaining solutions. As Cooper, Heirdsfield and Irons (1995 & 1996) reported, this has resulted in a tendency for Queensland children to use strategies for mental computation that reflect the procedures underlying the pen and paper algorithms regardless of their knowledge and ability to use more efficient strategies.

This paper reports on the progress of two children, Catherine and Adrien, in traditional classrooms over a period of five years. These children used a variety of creative strategies before traditional algorithms were taught to them.

THE STUDY

Subjects

Catherine and Adrien were two of 140 children chosen in Year 2 to participate in a large five year longitudinal Australian Research Council funded study into children's spontaneous mental strategies for addition and subtraction (Years 2 to 6). Children were originally chosen by the teachers to represent one third above average, one third average, and one third above average.

Catherine and Adrien attended different schools, Catherine attending a State Primary School and Adrien, a Catholic School, both being coeducational schools. Adrien changed schools in Year 5 and attended a Catholic boys' school. Although a focus of the study did not include classroom practice, anecdotal records indicated traditional teaching practices in both classrooms. Classroom discussion was rarely (if ever) heard. Children were only ever called upon to explain taught procedures. The teacher's voice was the predominant voice heard in the classrooms, as review of the videotapes testified. Blackboards were covered with many examples of algorithms, and rarely contained any contextual problems.

The children's ability levels were not revealed to the interviewer; however, their performance indicated that Catherine and Adrien were likely to belong to the "above average" group. Adrien's classroom teacher stated that he excelled in mathematics. However, Catherine's teachers stated that she was "not as good at maths as some make out".

Interview Procedures

The children were withdrawn from the classroom and the videotaped interviews were conducted in a separate room. The instrument was Piaget's clinical interview technique. The interview duration was limited to 30 minutes, to avoid the children's tiring. There were three addition and subtraction interviews in Year 2 and two interviews in each of Years 3 to 6. The children were presented with tasks, asked to solve them, and directed to explain their working.

Tasks

The tasks consisted of one, two and three-digit word problems and algorithmic exercises (see Cooper, Heirdsfield, & Irons, 1995 & 1996). The word problems were presented in number and picture form (the children listened as the interviewer said the problem). The addition word problems consisted of join addition; the subtraction word problems consisted of take away, missing addend, and difference. Algorithmic exercises were presented in both vertical and horizontal form. The numbers were chosen in the hope of eliciting wholistic strategies (c.f., traditional pen and paper algorithms); for example, 246 + 99 is more efficiently solved by employing the wholistic strategy of adding 100, then taking 1. Over the period of research, progressively more difficult examples were presented, as the children matured. There was an attempt to present the easier examples throughout all the interviews, but this was not always possible because of time constraints on the length of the interviews.

FINDINGS

In the literature, computational strategies have been classified (e.g., Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995) into 5 major categories (see Table 1). It is the intention of this paper to report detailed descriptions of interesting strategies and not to collapse full and interesting descriptions into single terms. However, the categories are useful for comparing children and for describing strategy change across time.

Strategy Use Across the Interviews

In Table 2 the more interesting solutions for particular questions are reported for each child across the interviews. The last two interviews in Year 6 are not reported in the table, as no new strategies were used by the children, and one predominant strategy was being used by both children. This phenomenon is discussed later.

Table 1

Mental Strategies for Addition and Subtraction

Strategy		Example
Counting		**28+35**: 28, 29, 30, .. (count on by 1)
		52-24: 52, 51, 50, .. (count back by 1)
Separation	*right to left (u-1010)*	**28+35**: 8+5=13, 20+30=50,63
		52-24: 12-4=8, 40-20=20, 28 (subtractive) : 4+8=12, 20+20=40, 28 (additive)
	left to right (1010)	**28+35**: 20+30=50, 8+5=13, 63
		52-24: 40-20=20, 12-4=8, 28 (subtractive) : 20+20=40, 4+8=12, 28 (additive)
Aggregation	*right to left (u-N10)*	**28+35**: 28+5=33, 33+30=63
	:	**52-24**: 52- 4=48, 48-20=28 (subtractive) 24+8=32, 32+ 20=52, 28 (additive)
	left to right (N10)	**28+35**: 28+30=58, 58+5=63
		52-24: 52-20=32, 32-4=28 (subtractive) : : 24+20=44, 44+8=52, 28 (additive)
Wholistic	*compensation*	**28+35**: 30+35=65, 65-2=63
		52-24: 52-30=22, 22+6=28(subtractive) : :24+26=50, 50+2=52, 26+2=28 (additive)
	levelling	**28+35**: 30+33=63
		52-24: 58-30=28 (subtractive) : 22+28=50, 28 (additive)

One digit word problems were solved by recall or by using the *build to 10* strategy by Adrien e.g., [14-8=(14-4)-4], and Catherine [e.g., 8+5=(8+2)+3]. Catherine continued to use the strategy for more difficult calculations [e.g., 136+56=(130+50)+(6+4)+2], as did Adrien (e.g., 75-29 in Interview 1, and 136+56 in Interview 8, see Table 2).

Table 2

Addition and Subtraction Strategies for Word Problems and Algorithmic Exercises, used by Catherine and Adrien

Interview	Catherine's Strategies	Adrien's Strategies
1 (Term 1, Year 2)	**45-21**: (45-20)-1=25-1=24.	**45-21**: (45-20)-1=25-1=24. **32+15**: 32+10, 42+5=47. **29+35**: 25+35, 60+4=64. **36+29**: 36+9, 45+20=65. **75-28**: 75-20, 55-5, 50-3=47. **96-39**: 96-30, 66-9=57. **65-38**: 38+22=60, 5 more to 65, 22+5=27. **95-26**: 95-20, 55-6=49.
2 (middle Year 2)	**75-28**: 70-20, 50-8, 42+5=47. **96-39**: 90-30,60-10, 50+7=57.	
3 (Term 4, Year 2)	**75-28**: 70-20,50-3(3 below)=47.	
4 (Term 2, Year 3)	**38+37**:30+30, (8+2)+5, 60+15=75. **136+56**: (((130+50)+6)+4)+2. **100-59**: 50+50=100, 50-9=41.	**130-49**: 130-9, 121-40=81. **253-98**: 250-90,160-10, 150+5=155.

Interview	Catherine's Strategies	Adrien's Strategies
5 (Term 4, Year 3)	**95-47**: 90-40, 50-2=48. **106-88**: (100-80)-2=20-2=18. **90-68**: 6+3=9(90), 30-10=20, 20+2=22. **165+217**: 200+100, 60+10, 5+7. **250-127**: 200-100, 50-20, -7.	**136+56**: 136+50, 186+6=192. **246+99**: (246+100)-1=345. **246+178**: 246+170, 416+8=424. **82-57**: 57+23=80,80+2=82, 23+2=25. **253-98**: (253-100)+2=155.
6 (Term 1, Year 4)	**106-88**: 80+20=100, 20-2=18. **253-98**: 98+2=100, to 253 is 153, 153+2=155. **100-59**: 50+50=100, 50-9=41. **96-49**: 5+4=9 (90), 50-3=47. **325-168**: ((300-100)-40)-3.	**106-88**: 106-6=100, 100-82=18. **253-98**: (253-100)+2=155. **246+99**: (246+100)-1=345. **246+178**: 240+170=410, 410+6=416, 416+8=424. **92-57**: 92-60, 32+3=35. **62-36**: 62-30, 32-6=26.
7 (Term 3, Year 4)	**106-88**:88+2=90, 90+10=100, 100+6=106, 2+10+6=18. **325-168**:3-1=2(200), 100, 12-6=6, 50, 15-8=7. **92-57**: 57+3=60, 60+30=90, 90+2=92, 3+30+2=35. **100-59**: (100-60)+1. **120-45**: (120-20)-25. **253-98**:98+2=100, 100+100+3=253.	**106-88**: 106-16, 90-2=88. 16+2=18. **325-168**: 300-100=200. 100-68=32 and 25+32=57, answer 157. **92-57**: 92-30=62, 62-5=57, 30+5=35. **62-36**: 60-36, 24+2=26. **400-337**: ((400-300)-30)-7=63. **250-127**: 250-130, 120+3=123.
8 (Term 1, Year 5)	**79+45**:7+3=10,10+1=11 (110)9+5=14, 110+14=124. **95-47**: 47+40=87, 87+8=95, 40+8=48. **106-88**: 88+20=108, 108-2=106, 20-2=18. **253-98**:98+2=100, 100+53=153, 153+100=253, 2+153=155. **246+99**: 299 goes to 300, 300+45=345. **400-337**: ((400-300)-30)-7=63. **325-168**: 168+40=208, 208+100=308,308+20-3=325, 40+100+17=157.	**79+45**: 79+21=100, 100+24=124. **95-47**: (95-45)-2=40-2=38. **106-88**: 100-88=12, 12+6=18. **253-98**: (250-98)+3. **246+178**: (250+180)-6=430-6=424. **136+56**: 135+55, 190+2=192. **92-57**: 57+30=87, 87+5=92, 30+5=35. **62-36**: 62-2, 60-36, 24+2=26. **250-127**: 250-130, 120+3=123.
9 (Term 3, Year 5)	**100-59**:60+40=100, 40+1=41. **95-47**: 47+40=87, 87+8=95, 40+8=48. **92-57**:57+30=87, (87+3)+2=92, 30+5=35. **120-45**:45+60=105, 105+15=120, 60+15=75. **106-88**: ((88+2)+10)+6=106. 2+10+6=18. **400-337**:(337+60)+3=400. 60+3=63. **253-98**:98+2=100, 100+153=253, 2+153=155. **246+99**: 299 goes to 300, 300+45=345.	**100-59**: (100-60)+1=40+1=41. **95-47**: 100-47, 53-5=48. **92-57**: 67, 77, 87, +5, so 35. **120-45**: (100-45)+20=55+20=75. **106-88**: (106-90)+2=16+2=18. **400-337**: (400-340)+3=60+3=63. **79+45**: 79+50, 129-5=124. **253-98**: 255-100=155. **250-127**: 250-130, 120+3=123

Strategy Descriptions

Catherine showed some understanding of negative numbers. In Interview 3, she solved 75-28 by, "70 take 20 is 50. 3 below 50 is 47 (5 take 8 is 3 below)"; again for 95-47 and 106-88 in Interview 5; for 96-49 in Interview 6, and interestingly for 325-168 again in Interview 6. Adrien tended to use the whole number in his calculations (e.g., for 29+35 in Interview 1, for 253-98 in Interview 5, through to 253-98 in Interview 9). Adrien did not employ a *wholistic* strategy for the solution of 253-98 in Interview 4, but his strategy was nevertheless advanced for a child in Year 3. Catherine also used the whole number in some of her calculations (e.g., 106-88 in Interview 5, 100-59 in Interview 7), but not as consistently as Adrien.

Catherine tended to progress through calculations, left to right, whether she separated the numbers into place values or left one number as a whole (e.g., separate into place values - *separation*: 136+56: 130+50, 180+6, 186+4, 190+2; leave one number as whole - *aggregation*: 95-47: 47+40=87, 87+8=95, 40+8=48). Adrien also tended to calculate in a left to right fashion (*separation* and *aggregation*), when not using a *wholistic* strategy (e.g., 32+15: 32+10=42, 42+5=47). Although the children had no difficulty employing a left to right strategy, even when regrouping was employed, it became glaringly evident that some of the teachers did not understand its use, as the following example illustrates. When one of the researchers was explaining the purpose of the research to some of the teachers of the children who participated in the interviews, when the left to right procedure was mentioned, one of the teachers interjected with, "but you can't do it that way, because it's wrong!" Not only did the children use that method, but they understood it and were accurate. Further, left to right procedures have been reported elsewhere in the literature (e.g., Madell, 1985).

It was interesting to observe the changes in strategies for particular questions over the interviews. An example of changing strategies can be seen in Adrien's solutions to 253-98 over Interviews 4, 5, 6 (5 and 6 the same), and 8 and 9. The strategy that Adrien employed was mostly *wholistic*, but with variations. It could be argued that the strategy Adrien employed in Interview 9 was the most advanced, as it required a great deal of understanding of number and operations, yet involved less burden on working memory. Other examples were solved more consistently, for example, Catherine's solutions for 253-98, which was solved additively (although there were slight variations).

It was also interesting to note that new strategies arose over time, and a particular strategy was used for more than one example in the one interview. An example of this is the additive, build up strategy Catherine used for subtraction examples (106-88, 400-337, and 253-98) in Interview 9. A different additive strategy was also used by Catherine in Interview 9 for the examples, 95-47, 120-45, and 92-57. Further, a similar strategy was used by both children for the same example in the same interview, for example, 45-21 in Interview 1. Also, both Adrien and Catherine used the same strategy to solve the same example (400-337), but in different interviews (Catherine in Interview 8 and Adrien in Interview 7).

It should be noted, however, that children did not use a consistent strategy for all questions of the same type in any one interview. For instance, even in Interview 1, Adrien did not use the same subtraction strategies to solve all the subtraction examples. Why he used different strategies remains a mystery. The same strategy could have been used for 75-28, 96-39, and 65-38, but this was not the case. Similarly, one would surmise that 29+35 and 36+29 (in Interview 1) would have been solved employing the same strategy, but this was certainly not the case (see Table 1). It appears that Adrien had a variety of strategies at his disposal, even at such an early age.

As the children progressed through the years, a strategy similar to the taught pen and paper algorithm was used by both children. By Interview 7, Catherine was solving 246+99 by, "9+6=15, carry the 1, 1+9+4=14, carry 1, 2+1=3, answer is 345." Yet previously in Interview 4, she solved the same example by, "(240+90)+(9+1)+5". The method used in Interview 4 would appear to require less load on working memory, but it was not her chosen method in the later interview. For algorithmic exercises in the first 6 interviews, Catherine separated the numbers into their place values and proceeded left to right. However, in Interview 7, she calculated right to left, the method used in the classroom. Adrien's solution method for 246+99 of adding 100, then subtracting 1 was consistent from the final interview in Year 3 to the final interview in Year 6. However, by Year 6, Adrien was employing the strategy similar to the taught pen and paper algorithm for other examples, particularly the algorithmic exercises. It appears, then, that although these two children were capable of employing efficient and effective strategies for computation (even before classroom instruction of pen and paper algorithms had taken place), by Year 6, both children were using the less efficient (i.e., mentally inefficient) written algorithm. Thus, there was an instructional effect.

When Adrien commenced at the new school in Year 5, all boys were pre tested in subject areas, and placed in streamed classes. It was interesting and a little perplexing to discover that Adrien was not placed in the top group. It is quite possible that Adrien's excellent number understanding did not carry through to other mathematical areas. However, a little more was revealed when another boy who had also come from another school and was now at the present one, was streamed into a far lower group, yet, he too had demonstrated good number understanding during interviews. He came to one interview in tears, because he had been marked wrong in a test, when he had copied an example incorrectly, but had managed to solve his example successfully. It appeared that the new school demanded correct copying and "correct" procedures.

Other Strategies

Apart from the variety of strategies employed by Catherine and Adrien, a large variety of strategies was also documented from the large study. As an example, Table 3 shows a variety of strategies employed for the example, 79+45 (word problem).

Table 3
Strategies Employed to Solve 79 + 45

79 + 45:	
• ((79 + 21) + 4) + 20	• ((70 + 40) + 9) + 5
• (79 + 50) - 5	• (70 + 40) + (9 + 5); 110 + 14
• 80 + 44	• (70 + 40) + (10 + 4); 110 + 14
• (80 + 40) + 4	• (9 + 5) + (70 + 40); 14 + 110
	• 9 + 5, carry 1, 7 + 4 + 1

These strategies cover all those mentioned in Table 1 - *separation*, *aggregation*, *wholistic*, and *mental image of pen and paper algorithm* (with the exception of *counting*). The strategies were employed at different times by different children. Further, the only strategy that was taught in any classroom was the last one.

CONCLUDING COMMENTS

Both Catherine and Adrien exhibited understanding of place value and effects of operations on numbers. It was also obvious that the two children thought about the numbers as wholes, rather than as single digits in columns. They also possessed a sound knowledge of number facts. Even when fact recall was not used, advanced facts strategies were employed, for example, *through 10*, which was used by both children. One final observation is that the children had access to a variety of strategies both during interviews and over the interviews that were reported here, that is, they exhibited flexibility.

Although it is clear that some children are capable of formulating their own efficient and varied computational strategies, I am not asserting that all children should be able to do this at such an early stage. It is obvious many children struggle with number concepts for many years. I am also not asserting that all children should be taught mental strategies of the calibre that Adrien and Catherine were able to formulate. After all, both children formulated different strategies and at different times, and with differing levels of complexity. However, young children should be given the opportunity to develop understanding and be encouraged to create their own computational methods, a proposal supported by such researchers as Carroll (1997), Kamii, Lewis, and Livingston (1993), and Madell (1985).

The answer does not lie in teaching mental strategies. At present, formal written algorithms are taught, and yet, children are still far from successful (McIntosh, 1991; Carraher, Carraher, & Schliemann, 1987). Further, there is little point in teaching algorithms, as "it is hard to follow the reasoning of others. No wonder so many children ignore the best of explanations of why a particular algorithm works and just follow the rules" (Madell, 1985). Although the traditional pen and paper procedures are efficient for some complex examples that may require calculations to be completed on paper, they are hardly efficient for examples of the type, 246+99 and 253-98, where wholistic strategies employed mentally or with pen and paper are far more efficient. Further, very complex examples are better computed with the use of a ten dollar calculator.

The research reported in the beginning of this paper supports recommendations for change in the mathematics classrooms. The concluding comments of this paper will summarise some of the insight learnt from these studies.

Children should be encouraged to invent their own computational procedures, as they develop better understanding of the effects of operations on number, and place value. Further, children take responsibility for their own learning. Time should be spent on students' describing various solution strategies for problems, and these strategies should be valued. Children's discussions are useful for not only discovering their understandings, but also any misconceptions. Finally, pen and paper calculations of the type, 27+5, 32-3, 100-1, and 102-97 should be avoided. Such calculations are more efficiently completed mentally.

In conclusion, Catherine and Adrien were members of very traditional classrooms. Yet, they were able to avoid the trap of resorting to taught mechanical algorithms in their mental calculations. They often employed far more efficient strategies, ones that exhibited number sense. What would they have been able to do if they were members of one of the "reform" classrooms?

REFERENCES

Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education, 24*(4), 294-323.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C., & Loef, M. (1989). Using knowledge of children" mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499-531.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1987). Mathematics in the streets and in schools. *British Journal of Developmental Psychology, 3*, 21-29.

Carroll, W. M. (1997). Mental and written computation: Abilities of students in a reform-based curriculum. *The Mathematics Educator, 2*(1), 18-32.

Cooper, T. J., Heirdsfield, A., & Irons, C. J. (1995). Years 2 and 3 children's strategies for mental addition and subtraction. *Mathematics Education Research Group in Australasia, 1*, 195-202.

Cooper, T. J., Heirdsfield, A., & Irons, C. J. (1996). Years 2 and 3 children's mental strategies for addition and subtraction money problems. In J. Mulligan & M. Mitchelmore (Eds.), *Children's number learning* (pp. 147-162). Adelaide: Australian Association of Mathematics Teachers.

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162.

Kamii, C., Lewis, B. A., & Livingston, S. J. (1993). Primary arithmetic: Children inventing their own procedures. *Arithmetic Teacher, 41*(4), 200-203.

Madell, R. (1985). Children's natural processes. *Arithmetic Teacher, 32*(7), 20-22.

McIntosh, A. (1991). The changing emphasis in arithmetic in the primary school: From written computation to mental computation and calculation use. *Journal of Science and Mathematics Education in S.E. Asia, 14*(1), 45-55.

Olivier, A., Murray, H., & Human, P. (1990). Building on young children's informal arithmetical knowledge. In G. Booker, P. Cobb, & T. N. de Mendicuti (Eds.), *Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education with the North American Chapter, 12th PME-NA Conference. Volume 3.* (pp. 297-304). Mexico: PME.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. E. Glaser (Ed.). *Advances in instructional psychology. Vol. 3.* (pp. 41-95). Hillsdale, NJ: Lawrence Erlbaum Associates.

Reys, R. E., Reys, B. J., Nohda, N., & Emori, H. (1995). Mental computation performance and strategy use of Japanese students in grades 2, 4, 6, and 8. *Journal for Research in Mathematics Education, 26*(4), 304-326.

Thompson, I. (1994). Young children's idiosyncratic written algorithms for addition. *Educational Studies in Mathematics, 26*, 323-345.

Acknowledgement

The author wishes to thank her co-researcher Tom J Cooper for his support in developing this paper. The paper is based on research supported by ARC grant No A79030404.