ASSESSMENT OF STUDENTS' UNDERSTANDING IN GEOMETRY: THE DIFFICULTIES IN WRITING GOOD QUESTIONS

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This paper focuses on the difficulties encountered in writing good questions for the assessment of students' understanding in geometry. A set of questions considered suitable for use in the classroom as a written test, were trialed on 106 Year 10 students. Following assessment of the students' responses, the questions were analysed with regard to their ability to elicit statements demonstrating students' understanding in geometry.

INTRODUCTION

There is need to be able to assess concurrently students' understanding in geometry. The availability of a reliable assessment procedure provides a teacher with information on the effectiveness of the education as well as providing a tool for teachers to provide evidence and feedback on what students know, understand and can do. Lawrie (1998, p.54) observed that clinical interviews, while giving an in-depth knowledge about students' understanding, are very time consuming, and, hence, are not particularly suitable for assessing large numbers of students. English (1998) emphasised that it is important that accuracy is not equated with understanding. Her study demonstrated the need for a range of questions designed to reveal a student's understanding. As interest in the application of mathematical knowledge, strategic knowledge, higher-order thinking, and problem solving become more prevalent, there is a concurrent need for assessment instruments that provide measures of these processes. The nature of mathematics and pedagogical approaches for teaching mathematics warrant consideration of specific assessment techniques in the area of mathematics. For example, deductive proof is prominent in mathematics in establishing truth (Webb, 1992, p.671). If Webb's (p.667) approach is adopted, i.e., that the method of assessment should depend largely on the purpose of assessment and the information that is to be gleaned from it, then there is need for a set of questions suitable for use in a classroom situation as a written test, the responses to which provide the teacher with a measurement of the degree of understanding of the topic by the students.

A set of items (known as the UGAT test (Understanding in Geometry Assessment Test)), suitable for testing for a student's van Hiele level of understanding of the square has been produced at the University of New England by Lawrie and Pegg (in press). This paper reports on the difficulties encountered in writing the UGAT items.

BACKGROUND

The Nature of Test Items

Item content and structure is not all that important asserted Biggs and Collis (1982, p.52), provided that the task set, or question asked, allows higher-level responses to be considered relevant. For example, a question that begins 'List the various factors that' is setting a van Hiele Level 2 criterion. On the other hand, 'Compare and contrast' invites a van Hiele Level 3 response.

Smith (1989) made a comparison between two geometry tests, a modified version of Usiskin's (1982) multiple-choice test and the written test developed by De Villiers and Njisane (1987), in which the nature of the questions ranged from simple yes/no items to those requiring construction of a formal proof. While both tests were found to be useful for the allocation of students' van Hiele levels, the open-ended type questions gave more consistent results than did the multiple-choice questions.

Ellerton and Clements (1997, p.157) reported that in their research, designed to test the effectiveness of multiple-choice and short-answer questions,

> over one third of responses ... were such that (a) correct answers were given by students who did not have a sound understanding of the correct mathematical knowledge, skills, concepts and relationships which the questions were intended to cover; or (b) incorrect answers were given by students who had partial or full understanding.

Hence, questions suitable for the UGAT test would be open questions which invited higher-level responses and which encouraged students to display their understanding in geometry. Responses to the questions would be assessed for the van Hiele level of understanding in geometry displayed therein.

Van Hiele Level Descriptors

A brief description of the first four van Hiele (1986) levels, the ones commonly displayed by secondary students, and, hence, the most relevant to this study, is given:

Level 1 A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.

Level 2 The figure is identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

DESIGN

The study was designed in four stages:

1. the selection of several questions, each investigating understanding of the concept square;
2. trial of the initial set of questions on Year 10 students;
3. discussion of the ability of each test item to ascertain a student's van Hiele level of understanding based on the analysis of the students' responses to that item; and
4. production of a final and smaller set of test items to be known as the UGAT test for the concept square.

In selecting appropriate questions, several factors were considered. Each question was phrased so as to probe for a statement indicating the depth of the student's reasoning in geometry. While responses at van Hiele Levels 1 and 2 were be short answers identifying figures or listing knowledge about properties, responses at Levels 3 and 4 would include a justification of the knowledge displayed. Hence, questions probing for expression of Levels 3 and 4 reasoning should include phrases such as 'How do you know?', 'What can you say about ... ?', 'Why?', 'Give your reasons.', 'Compare and contrast ...'. Such questions were considered suitable for inclusion in the first UGAT test assessing understanding of the square.

Since the focus of the NSW Years 9 and 10 syllabus is no higher than early Level 4, the questions were to test for understanding of the first four van Hiele levels. They consisted of items designed by, or derived from those used by Burger and Shaughnessy (1986) and Mayberry (1981), together with questions designed by Lawrie and Pegg. All questions except one were to have the potential to be answered at more than one van Hiele level. The exception was a question designed solely to determine whether a student could select the squares from a group of figures (Level 1). The final trial selection was a set of fifteen items able to be completed within a forty-minute lesson.

The trial UGAT test was given to 106 Year 10 students from three secondary schools in Armidale, one government school, one systemic Catholic school and one private school. In Years 9 and 10 in NSW, mathematics is taught at three levels, Advanced, Intermediate and Standard. The 106 students came from six classes which represented all levels, three Advanced classes, two Intermediate classes, and one Standard class. Each student response was assessed for the van Hiele level displayed therein.

The responses to each question were analysed to evaluate whether that question had the potential to elicit statements expressing understanding in geometry. This involved an evaluation of the ease of coding the students' responses to a question, together with the consistency of the responses in relation to the students' other answers. A final set of questions known as the UGAT test (square) was then formulated.

DISCUSSION

Although all questions selected for the trial test paper were considered to have the potential to elicit responses which could be assessed for understanding, not all questions provided the expected information. In particular, it was found that the responses to some questions did not, by themselves, give the required information. Following analysis of the students' responses, the questions were categorised into five groups. These are described in Table 1. Questions in Groups 1 and 2, and the amended version of those in Group 3 were considered suitable for the UGAT test. Questions in Groups 4 and 5 were found to be unsuitable questions.

Table 1
Classification of Question Types

	Description of questions	Questions
Group 1	questions which elicited responses exhibiting the expected van Hiele levels	2, 10 and 14
Group 2	questions, the answers to which failed to represent all the expected van Hiele levels but otherwise produced assessable responses	3, 6 and 15
Group 3	questions, the responses to which indicated need for some modification in either the phrasing or the design	1, 5, 8, 11 and 13
Group 4	questions whose responses could be categorised only with consideration of a student's responses overall	4 and 12
Group 5	questions which elicited mainly rote, recalled or prompted responses, and, hence, were considered not assessable	7 and 9

Discussion of Types of Questions

Each question type is discussed through an analysis of the responses to relevant questions.

Group 1

Q.14 was designed to test whether students have understanding of necessary and sufficient conditions in a definition (Level 4), appreciation of a definition (Level 3), or awareness of properties (Level 2). Responses to the last two parts of the question, 'Which are the least clues necessary to identify the figure?' and 'Why?' are the main indicators of the depth of a student's reasoning.

Analysis of responses to Q.14 showed it to be a question which encouraged students to demonstrate the depth of their understanding. All except five students attempted the question, and all responses were able to be coded for a van Hiele level of understanding. A weakness in the question lies in the prompting of earlier questions that the figure is a square. However, this weakness is countermanded by the following explanations. Students gave responses reflecting all three van Hiele Levels. Examples of typical responses to the final part 'Why?' are given for the three levels.

Question 14

a. I am a quadrilateral.
b. My diagonals are perpendicular bisectors of each other.
c. One angle is a right angle.
d. All angles are right angles.
e. Opposite sides are parallel.
f. Opposite sides are equal.
g. Adjacent sides are equal.

What am I?
Which are the least clues necessary to identify the figure?
Why?

N04 *To show that it has 4 sides (a), and if the diagonals are perpendicular bisectors then all 4 sides must be equal (b). (c) to show that it is not a rhombus.* (Level 4, applying necessary and sufficient conditions.)

N02 *because quadrilateral says it has four sides, all angles at right angles means it must be some type of rectangle and adjacent sides equal means that all sides must in this case be equal.* (Level 3, showing understanding of implications of properties, but including too many conditions.)

O07 *If all angles are right angles, there can only be four sides and if all sides are equal, it must be a square* (Level 2, listing of properties.)

Two interesting patterns were perceived in the answers to this question. First, in response to 'Which are the least clues necessary to identify the figure?', the low achievers tended to write in full all conditions whereas the high achievers generally listed the code letters only. Second, many of the students giving Level 2 answers left a blank for the last question part, finding the final question part as superfluous.

Group 2

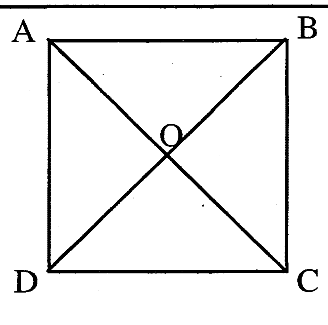
Responses to Q.3 should determine whether a student can identify a named angle and give its size (Level 2), and then explain how they know (Level 2 and (maybe) Level 3).

Question 3

ABCD is a square.

∠ CBO = ______°
How do you know?

∠ AOD = ______°
How do you know?



The question was expected to elicit responses representing van Hiele Levels 2 and 3. However, on assessing the responses, none were found to reflect Level 3 reasoning. It is considered that the structure of the question fails to encourage students to look for relationships between properties. Hence, the expectation of Level 3 responses is considered unrealistic. To assess students as not having Level 3 understanding for this question is not reasonable, consequently, this question is considered suitable for inclusion solely as an assessment of Level 2 knowledge.

All except five students gave assessable responses. Two responses were considered to demonstrate early sequencing, but not of sufficient complexity to warrant a Level 3 grading.

N06 *$\angle CBO = 45°$. The axis-of-sym[m]etry cuts the 90° angle exactly in half. ∴ it is 45°*

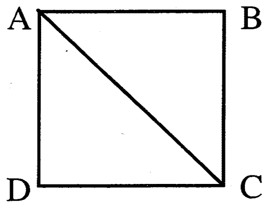
O86 *$\angle BOC$ is a right angle. 180° - 90° = 90°. The two angles are equal so $\angle CBO = 45°$*

Many low achievers interpreted $\angle$CBO as meaning 'the angle sum of ΔCBO'. This elicited many responses of "180°, because the angles of a triangle add to 180°."

Group 3

Q.13 contains a proof using congruency, in which the reasons for each step of the proof were to be completed together with a final statement of what had been proved. The responses to the question were expected to identify to what degree students could sequence their reasoning (Level 3). Level 2 responses were not expected to include acceptable statements for the last two parts.

Question 13
Complete the missing steps for this proof.



ABCD is a square

In triangles ADC, ABC,
AD = AB (equal sides of a square)
CD = CB (____________)
$\angle$ADC = $\angle$ABC (____________)
Therefore the triangles are congruent (____________)
Therefore $\angle$DAC = $\angle$BAC,
and $\angle$DCA = $\angle$BCA (____________)

What has been proved?

Results were encouraging, with ninety-three students giving assessable responses. It was found to be easy to distinguish between Level 2 and Level 3 responses, and to grade the depth of understanding displayed in each answer within the level. However, several students giving Level 2 responses found the given reason in the first line of the proof '(equal sides of a square)' a useful prompt, and so completed the next two lines of proof with "(equal sides of a square)" and "(equal angles of a square)". They seldom made any further statements. Hence, it appears that such a prompt may, in fact, inhibit the thinking of some students. It was decided that if the prompt in the first line of the proof were omitted, Q.13 would be a better question.

This question has now been amended twice. Originally, the question was designed to end with a concluding statement 'Therefore the diagonal bisects the angles of the square.' instead of the question, 'What has been proved?' However, it was recognised that in such a form, most of the question could be completed with recalled facts, and, therefore, is of little value.

More responses to this question (those of seventeen students), were considered to reflect Level 3 understanding than for any other of the trial questions. This is considered to reflect the amount of attention given in the classroom to congruency proofs.

Group 4

The intention of Q.4 is to differentiate between students reasoning at van Hiele Levels 4, 3 and 2. A Level 4 response is (b) and (e), necessary and sufficient conditions for a square, while a Level 3 response would be likely to include too many details, showing necessary but not sufficient conditions. A student responding at Level 2 circles as many properties as is known, often excluding both (a) and (b).

Question 4
Circle the smallest combination of the following which guarantees a figure to be a square.
a. It is a parallelogram.
b. It is a rectangle.
c. It has right angles.
d. Opposite sides are parallel.
e. Adjacent sides are equal in length.
f. Opposite sides are equal in length.

It was found that very few responses to the question could be categorised without examining the student's responses to all other questions. For example, a response of (b) and (e), as well as being a Level 4 response, was found to come from a Level 3 student attempting to minimise conditions without understanding the implications. The same response was given also by a student whose responses to all other questions reflected van Hiele Level 2 understanding. Another response, selecting (b), (d), (e) and (f), was given by a large number of students. Some of these students had demonstrated Level 3 reasoning in several questions, while others had responded to all questions with Level 2 answers. It was not possible to code this answer without recourse to a student's overall performance. Compounding this difficulty in assessing the level displayed in responses were two extra factors, namely, the attempts by some students to minimise their response, and the prompting factor resulting from the listing of many properties.

As a consequence of the above analysis, it is considered that Q.4 is not a good question for ascertaining a student's understanding in geometry.

Group 5

Two questions, Q.7 and Q.9, which were expected to give information about students' understanding in geometry, were found to elicit responses that appeared to be mainly recalled or prompted statements. Hence, the questions were considered not to generate sufficient assessable information.

Question 7
Will two squares always be similar? YES NO
Give reasons for your answers.

In Q.7, the reasons for a yes/no answer were expected to reveal whether a student understood the implications of the angle and/or side properties of a square (Level 3). Responses were of the kind "all angles are equal", "they will always be the same shape (different size)" and "the sides are in proportion". While only five students failed to give a reason and nine students commented on the proportionality of the sides, the remainder of the students (ninety-two) focused almost equally either on the equality of the angles or on the shape being the same but the sizes different. It was considered the responses did not indicate sufficiently the amount of understanding of the students.

Question 9
ABCD is a parallelogram.
AB and BC are equal.
$\angle$ ABC is a right angle.

What have we proved?

Responses to Q.9 were expected to show whether a student understood the essence of a proof. It was considered that students reasoning at van Hiele Level 3 would show appreciation of sequencing but not an overview of the proof, while students reasoning at Level 2 would select a single statement from the proof.

While no responses were considered to indicate Level 4 understanding, a few students demonstrated Level 3 reasoning in their responses. Many students attempted to draw a diagram of the figure, and very many stated that the figure was a square, probably prompted by the other questions in the paper. The responses were grouped into the following types of answers.

Responses considered assessable :

a) a diagram together with some (incomplete) explanation of reasoning

b) a statement showing some reasoning; this was often a one-step development, combining a property of a parallelogram together with a piece of the given information, e.g."opposite angles are equal, therefore $\angle$ADC is also a right angle"

c) a diagram correctly showing given information, with or without a bare statement "This is a square, rectangle or rhombus"

Responses that were considered to be non-assessable:

a) no answer and nonsense statements

b) incorrect diagrams (such as a triangle) and diagrams of a standard parallelogram with no markings and no statement

c) statements solely repeating information given in the question

d) responses: "ABCD is a square"

The majority of responses (two-thirds) were found to be not assessable.

The weakness in this question lies in its design. There were very few steps (three) in the proof, and there is no request for justification of a student's statement. The most interesting responses were those containing students' attempts to draw the figure. Many of these students started with a standard parallelogram, then modified or drew over it to arrive at the figure of a square. Often, they appeared not to know how to interpret what they had drawn. Most of these students would be reasoning at van Hiele Level 2. While Q.9 could be amended so as to elicit a greater number of assessable responses, it is considered that Q.13 discussed above provided the same information in a more suitable format.

CONCLUSION

Whereas fifteen questions were designed to specifications described in this paper, analysis of the students' responses indicated that not all the questions were able to elicit the expected information about students' understanding in geometry. Six questions were found to provide responses which indicated a student's van Hiele level of understanding, although responses to three of these were not across as many levels as expected. Responses to a further five of the questions indicated that, with amendment, these questions should elicit responses giving the desired insight into student understanding. The remaining four questions were considered not suitable in that responses to them did not provide the researcher with sufficient information with which to assess students' van Hiele levels of understanding satisfactorily. In summary, good questions for the purpose of ascertaining students' van Hiele levels of understanding in geometry by means of a written paper do not result automatically from the application of a design formula.

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