HOW TO TEACH GENERALISATIONS IN MATHEMATICS?

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This paper is in three parts. In the first part, I compare and contrast three typical methods of teaching generalisations in mathematics. In the second, I describe the theory of generalisation expounded by the Soviet psychologist Vasily Davidov. In the third part, I re-examine the three methods of teaching mathematical generalisations in the light of Davidov's theory, and make some general conclusions.

For some time, Paul White and I have been promoting the importance of the abstraction process in mathematics learning (Mitchelmore & White, 1995; White & Mitchelmore, 1996). This approach is a reaction to decontextualised mathematics teaching. It is based on the assumption that abstract mathematical generalisations are reached by searching for common elements in a range of *several* concrete situations. On the other hand, we also believe in the problem-solving approach, where a student's struggle with *one* problem may be worth twenty done as exercises following a method given by a teacher. How are these various methods of learning and teaching related?

THREE TEACHING METHODS COMPARED

The ABC Method

What I call the ABC method—teaching **A**bstract **B**efore **C**oncrete—is widely used in mathematics teaching. The theory is that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). Steinbring (1989) shows that many school mathematics teachers believe that procedures should be learnt before applications, and textbooks and even curriculum guides often reflect the same order (Mitchelmore & White, 1995).

The ABC method has been widely criticised for leading (at the best of times) to *abstract-apart* knowledge—superficial knowledge that cannot be applied to problem situations and is quickly forgotten once examinations are over (Schoenfeld, 1988; White & Mitchelmore, 1996). Speaking of students who have been taught by the ABC method, Dreyfus (1991, pp. 28) writes: "They have been taught the products of the activity of scores of mathematicians in their final form, but they have not gained insight into the processes that have led mathematicians to create these products."

Empirical Methods

Diametrically opposed to the ABC method is the "concrete to abstract" approach espoused by many educators. In one manifestation of this approach, students are encouraged to model the pattern-seeking behaviour which is said to be fundamental to mathematics using concrete (manipulative or figural) materials. Two examples:

- Grade 4 students draw squares of dots: 1 by 1, 2 by 2, 3 by 3, and 4 by 4. They count the number of dots and write down the sequence obtained: 1, 4, 9, 16. They notice that these numbers go up by successive odd numbers (3, 5, 7) and infer that the next number must be 9 more than 16. Bingo! 25–it works.
- Grade 8 students draw a triangle, of whatever size and shape they like. The students then measure the three angles of their triangle. Answers vary around 180 degrees. The teacher calls this the "angle sum of a triangle theorem" and states that the angle is always 180 degrees (if you could measure it exactly).

Activities like these do show how new mathematical knowledge is often discovered, but the results are still mysterious. Some obvious questions are simply not asked: "*Why* do square numbers go up by odd numbers?" "Are you *sure* the angles in a triangle always add up to 180 degrees? Exactly?" If these questions are not asked, there is a constant danger of making false inductions. More importantly, it is answering the "Why?" questions which explains the results, and this is surely the essence of mathematics.

A Problem Solving Approach

Another reaction to the ABC method is to have students model another supposedly fundamental mathematical behaviour: problem solving. The tactic is to pose a problem and then leave students to solve it and convince others the solution is correct. An example I often use is: *Can you tessellate the plane using a scalene triangle?* (Mitchelmore, 1998, Activity P08). All sorts of geometry concerning congruence, angles, and parallels arise and become connected—in particular, angles on a straight line; the angle sum of a triangle; and corresponding, alternate, and co-interior angles formed by a line intersecting several parallel lines. In my experience, a period or two wrestling with this problem produces a far deeper understanding of these concepts and results than teaching each one separately (even using an empirical approach).

A Comparison: The Three Approaches to One Topic

The differences between the three approaches are best illustrated by considering how they would be used to teach one topic. This example was suggested by Boero and Garuti (1994), who describe a classroom investigation of indirect height measurement using shadows cast by the sun. All the Grade 8 students seemed to grasp the proportion method of calculating the inaccessible height, but most could not transfer the result to similar triangles in general. What should the teacher do next?

One method would be to present the theory of similar triangles, showing how this theory applies to height measurement and then applying it to other contexts. This is a variant of the ABC method. By presenting the theory after one application, students should at least be able to link the abstract concept of "similar triangles" to something concrete. But I predict that students would still not appreciate the generality of the concept, and would learn to manipulate problems in an abstract-apart fashion.

A second possible method is to study further contexts where similar triangles occur (e.g., scales, enlargements, gradients) and learn how to make the corresponding calculations in those contexts. Then, students can be led to notice the similarities between all such cases–between the objects studied, the relations between them, and the method of calculation. The common features can then be identified and abstracted: the objects are points, lines, and triangles; the crucial relation is similarity; and the method of calculation is proportion. This is a variant of the empirical method discussed above; the patterns that are sought in this case are much deeper, involving comparisons between relations, but the principle is the same. The concepts and relations abstracted would not be abstract-apart but *abstract-general* (i.e., general to many contexts).

A third method, a variant of the problem-solving method above, is suggested by Dörfler (1991). Take the result obtained from height measurement and ask the question: What conditions in the situation are *essential* to the result obtained? For example, the calculation involves two vertical and two horizontal lines; *must* the lines be vertical and horizontal? A short thought experiment (just rotate the page) is enough to see that the answer is clearly "no"–but at least the corresponding lines must be parallel. Must these two lines be perpendicular? This question requires some empirical investigation before coming to the same conclusion as for the first question. Must the third lines of the corresponding triangles be parallel? Again, empirical investigation is needed to reach the answer, "yes." Must all three corresponding lines be parallel, then? I can foresee much discussion and argument before coming to the conclusion that the essential condition is that the corresponding angles of the triangles be equal in size. This method of reaching a general result has much in common with the problem-solving approach outlined above. A long time is spent on one problem, but in the course of solving it many ideas are brought into relation and the *origin* of the resulting abstract concepts and the *value* of the resulting relationship is quite clear.

TYPES OF GENERALISATION

All three methods we have outlined are aimed at teaching mathematical *generalisations*– concepts and results which are generally true. The Russian psychologist Davidov has recently expounded two different types of generalisation using in teaching, including mathematics teaching (Davidov, 1972/1990). Although much of the book is full of Marxist jargon like *dialectical materialism,* there is a great deal of value in what he has to say. And since his work is not widely known in the West, I believe it is worth attempting to summarise and apply his theory. All page references below are to this book, unless otherwise stated.

Empirical Generalisation

Davidov summarises what he calls the traditional or *empirical* theory of generalisation as follows. A generalisation (also called a concept) arises from classification activity followed by "finding and singling out [properties] in a whole class of similar objects" (p. 10). The wider the range of objects or experiences included, the richer the concept. Generalisation is "inseparably linked to the operation of *abstracting*. Delineating a certain quality as a common one includes separating it from other qualities. This allows the child to convert the general quality into an independent and particular object of subsequent actions" (p. 13). The purpose of abstraction is thus to permit the application of general rules of operation in specific fields of application. Davidov argues that the processes of generalisation and abstraction are applied mainly to real objects, although they may later be applied to concepts derived from real objects. And he ascribes the pedagogical principle, "always proceed from the particular to the general," to a wide-spread belief in the psychological principle of empirical generalisation as the basis for all learning and teaching in school (not only mathematics).

The theory Davidov summarises here is essentially the same as that described by Dienes (1961), Piaget (1972), Skemp (1986), and others. While recognising that empirical generalisation has played an important role in the development of science and mathematics in the past, Davidov criticises it on several grounds:

- Classes are not formed by noticing common features. In fact, it is impossible to state the defining features of most everyday concepts (such as human). Also, classes are not formed from arbitrary collections of objects—there is always some reason for objects being grouped together (often a common function). Classifying on the basis of external characteristics does nothing towards identifying their inner connections.

- Teaching through empirical generalisation must consist of the transmission of concepts known to the teacher (who is aware of the inner connections) through examples chosen by the teacher which to the students appear to be unrelated. Also, the quality of a concept formed is determined by the objects in the class (again, chosen by the teacher) from which it has been abstracted.

- A restriction to empirical generalisation, by emphasising the link to concrete experience, discourages children from embarking on something which is qualitatively different: abstract mathematical thought.

Piaget does not escape Davidov's criticism. His theory is attacked firstly as being basically empirical and secondly as not proceeding beyond early adolescent thinking.

Implications for Teaching

Davidov cites a number of studies carried out by Soviet psychologists (mostly reported only in Russian) to support his criticism of empirical generalisation:

- Pototskii (1963) found that teachers often teach classification of problems instead of problem solving, as a result of which students cannot solve novel problems. Yaroshchuk (1957) found that nearly 90% of Grade 4 students solved those and only problems which they could identify as of a type they had solved previously.
- In one of his own studies (pp. 143-150), Davidov found that Grade 1 students who had mastered initial number concepts following the standard approach to teaching (judged by him to be based on empirical generalisation) were unable to transfer their counting skills to counting sets of sets of objects or measurement units.
- Krutetsky (1968) (also published in English in 1976) investigated 6th and 7th graders' ability to learn algebraic and other mathematical generalisations. He found that below-average students never learnt to generalise; average students learnt to generalise after many examples; and above-average students learnt to generalise after one or two examples. The below-average and the average students were apparently using empirical generalisation whereas the above-average students were using something else. Davidov interprets these results as demonstrating (a) the ineffectiveness and inefficiency of empirical generalisation, and (b) the debilitating effect of a method of teaching based on empirical generalisation.

Theoretical Generalisation

In a chapter entitled "Basic Propositions in the Dialectical Materialist Theory of Thought," much of which is taken up with references to the works of Marx, Engels, and Lenin, Davidov expounds the principles of what he calls *theoretical* or *content-based generalisation.* A few quotations should clarify the new way in which this theory uses the same terms as in empirical generalisation, plus some new ones:

> Generalization is achieved, not through simple comparison of the attributes in particular objects ... but through analysing the essence of the objects and phenomena being studied. (p. 295)
>
> To make such a generalization means to discover a principle, a necessary connection of the individual phenomena within a certain whole, the law for the formation of that whole. (p. 295)
>
> By *abstracting*, man isolates and ... mentally retains the specific nature of the real relationship of things (p. 294).
>
> Content-related abstraction and generalization underlie the formation of a scientific, *theoretical* concept. Such concept functions as a completely definite and concrete *method of connecting* the universal and the individual. (p. 296)
>
> Only when the origin of the object or a conception is clear to the student does it become possible to assert that ... the student has a concept of that object. (p. 334)
>
> The essence of a thing is none other than the basis (included in itself) for all of the changes that occur with it in interaction with other things. (p. 194)

We may note some particular aspects of these ideas:

- The emphasis on analysis, in place of comparison or similarity, as the means for identifying the concept to be abstracted. The term *content-based* emphasises that analysis can only operate on content, not on superficial appearances.
- The emphasis on connections (not only between elements but also between the general

and the particular) which exist for some reason, are necessary, the result of some law. It is analysis which brings out these necessary connections.

Terms like essence and concrete (also, elsewhere, internal, real, and true) correlate with the claim that content-based generalisation is deeply meaningful thought which "goes beyond the limits of sensory conceptions" (p. 300).

Implications for Teaching

Davidov cites several Soviet studies in support of the superiority of theoretical generalisation, including Krutetsky (1968/1976) noted above. Here are two more:

- Slavskaya (1958) showed that the detailed analysis of a basic geometrical problem led to immediate transfer to an auxiliary problem, students "singling out the essential link that connected it with the basic problem." Solving the auxiliary problem first, without any relation to the basic theorem, led only to a gradual generalisation "in the course of a detailed comparison of the features of both problems" (p. 201).
- Mashbit (1963) compared students who "ascertained initially the general structure of the solution method by analyzing particular problems which were models" with those who "solved particular problems in which the concrete conditions and the form for expressing a mathematical relationship varied." The second group made slow progress, whereas the first "travelled the path that is accessible only to the best prepared students" who used the second method (p. 329).

In proposing reforms to school mathematics teaching, Davidov (p. 320) argues that "there must be instruction ... that reproduces in compressed, abbreviated form the real historical process of the birth and development ... of knowledge. The child ... cannot independently 'acquire' what people have already attained, but he should *repeat* the discoveries of human beings in previous generations". He states the following general sequence for the introduction of new concepts using theoretical generalisation:

> 1) students' orientation in a problem situation ... whose solution requires a new concept, 2) mastery of a model for the sort of transformation of the material that discloses in it a relationship that serves as a general basis for solving any problem of the given type, 3) establishment of this relationship in an object-related or symbolic model, which permits its properties to be studied in "pure form," 4) disclosure of properties of the delineated relationship by which to deduce the conditions and methods of solving the original problem (p. 349).

He then reports briefly on an experimental primary school mathematics course which focuses on solving measurement and counting problems (in that order) and does not shy away from fractions and integers as they arise in such problems. Most remarkably, "For four years, the students solve all word problems only by setting up equations—that is, with no access to an arithmetical method. [Studies indicate that] this method, in the first place, is entirely accessible to children of 7-10 years ...; second, it substantially simplifies all of the instructional work on problems; third, it largely favors the children's development of skill in independently solving 'new' problems, one encountered *for the first time*" (p. 365).

APPLYING DAVIDOV'S THEORY

The Three Teaching Methods Classified

The three methods of teaching mathematical generalisations which we outlined in the first part of this paper can now be classified using Davidov's theory:

The "empirical method" is clearly based on the tenets of empirical generalisation. Results are recognised as properties which are common to several situations chosen by the teacher, but are not integrated into a general explanatory theory. Understanding may be superficial and the whole exercise pointless.

The "ABC method," although it is in a sense the opposite of the empirical method, is in fact also based on the theory of empirical generalisation. For the ABC method can be regarded as an attempt to short-circuit a lengthy process of empirical generalisation by presenting students with the abstract end product. Use of this end product in problem solving depends on recognising the problem as being of a type to which the abstract result may be applied–exactly as in the empirical method (but without the benefit of experience with several examples). Again, understanding is likely to be superficial and the purpose of the concepts studied obscure.

By contrast, the "problem-solving method" is clearly based on the principle of theoretical generalisation. Situations (tessellations, shadows) are analysed to identify the *structure* of the problem, the *basic* elements and the *essential connections* between them. The concepts which are formed in this way are not isolated, but take their meaning from their relations to one another.

Limitations of the Empirical Approach: One Last Example

The more I think about mathematics teaching, the more I see the limitations of empirical generalisation. Here is an example from my own textbook (Mitchelmore & Raynor, 1986), suggested by a similar case described by Peschek (1989):

In this textbook, linear relations are introduced using a double process of empirical generalisation as follows. The student is first set several arithmetical problems in familiar "linear" situations (taxi fares, electricity costs, etc.). From similarities between the various calculations within each problem, a general calculation procedure is formulated for each situation. Similarities between the calculation procedures in different situations are then identified: one quantity is always found by multiplying another quantity by a fixed number and adding another fixed number. The concept of *linear growth* is defined as such a relation. It is then found that the graph of linear growth is a straight line, and the algebraic form $y = kx + a$ is introduced.

As Peschek (1989) points out, the problem with this approach is that the essential characteristic of linear functions (that a given increment in the independent variable always leads to the same increment in the dependent variable) gets lost. It is always there implicitly, of course, but it is never made explicit. Students are never asked *why* each given situation leads to a linear function, or what types of situation are "linear." As a result, students do not learn a deep conception of linear function; instead, their concept is limited to the superficial form of the relation, essentially a procedural conception. And how is a student expected to judge whether a new situation is linear?

A Critique of Davidov's Theory

Although Davidov's theory seems to have much to recommend it, I can see some problems:

- Defining theoretical generalisation as the search for *essential* relations is unsatisfactory. How does one know what is essential? Also, what is essential in one context might not be essential in another.
- I cannot see how theoretical generalisation can take place without some empirical generalisation as its basis. For example, in generalising the shadows problem above, how does one find out that the condition that two lines be perpendicular can be relaxed to a requirement that corresponding sides be parallel, or that this condition can be relaxed to a requirement that corresponding angles be equal? Surely only by empirical investigation, not by theoretical reasoning.

- Moreover, an empirical investigation of relations can lead to an integrating structure just as a process of theoretical generalisation can. For example, the development of a language to express the communality between shadow, enlargement, and gradient situations can lead to an abstract formulation of the similar triangle properties which is at least as general as that obtained by an investigation of the essential elements in one of these situations.

CONCLUSION

Davidov's theory has much in common with several recent theories treating understanding in mathematics as "making connection between ideas, facts, or procedures" (Hiebert & Carpenter, 1992, p. 67) leading to the construction of what Ross & Hoyles (1996, p. 108) call *webbing*. I believe the theory would generously repay closer study by school and university teachers, and by teachers of teachers.

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