COMPUTER ALGEBRA SYSTEMS FACILITATE POSITIVE LEARNING STRATEGIES

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Refining mathematical understanding through the use of multiple representations and negotiation of meaning has been considered important for students. This paper reports on an observational study of students using a Computer Algebra System in an introductory calculus unit at undergraduate level. There was strong evidence that the use of this technology was a catalyst in students using these positive learning methods. However students felt that the use of this technology aided but did not underpin their learning of mathematics.

BACKGROUND

Computer Algebra Systems (CAS) provide symbolic manipulation, graphs and tables at the press of a button. It has been suggested (Mayes, 1993, Heid and Zbiek, 1995) that such packages may be used in the teaching and learning of mathematics to move the focus of learning from procedures to concepts. Day (1993) encouraged us that 'The power and flexibility of technology can help change the focus of school algebra from students becoming mediocre manipulators to their becoming accomplished analysts' (p30). Over the last decade, as this technology has become less expensive and more accessible, mathematics educators have started to explore the possibilities CAS may offer students.

Experimental studies have suggested that the use of CAS improved learning outcomes for mathematics students. Heid (1988) reported on a study undertaken with undergraduates studying an introductory calculus course. One group was taught using CAS to present and explore concepts while a comparison group was taught in a traditional manner without CAS. When assessed, both groups showed similar results on a skills test but the CAS group showed greater understanding of concepts and the ability to use different representations. Palmiter (1991) reported that an experimental CAS group had covered the same calculus course as a traditional group but with fewer hours teaching. The students in the experimental group outperformed the traditional group on both conceptual and computational examinations. These positive outcomes might be related to the learning methods fostered by the use of CAS.

A feature offered by Computer Algebra Systems is the ability to swap quickly and correctly between different representations ie between algebra, graphs and tables. Dreyfus (1991) wrote:

> To be successful in mathematics, it is desirable to have rich mental representations of concepts. A representation is rich if it contains many linked aspects of that concept...One does not get the support that is needed to successfully manage the information used in solving a problem unless the various representations are correctly and strongly linked. One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take....Teaching and learning this process of switching is not easy.(p32)

Using a CAS to produce different representations requires very little effort; this encourages students to swap between representations in order to find the information they require. This method of learning may help them develop Dreyfus' 'rich mental representations of concepts'.

Although there is not complete agreement about how mathematical concepts are learned, contemporary thought supports a view of 'constructivism'. This model recognises that all learning builds on experience. The mind naturally organises repeated experiences into complex networks of concepts, rules, and strategies referred to as schema. These schema are not fixed but continually change over time as students are exposed to examples and counter-examples (Romberg, 1993). One of the strengths of CAS is that they allow students to look at many examples (or non examples), represented algebraically, graphically, or numerically, in a short space of time.

A constructivist view of the learning of mathematics also emphasises that a student's understanding of concepts needs to undergo a process of personal and social negotiation before it is internalised (Vygotsky, 1978). Negotiation with peers, the interplay of ideas backwards and forwards, allows each student to refine their thinking based on the language and experience of others. Such a process may lead to a student 'assimilating' new information into their current schema or force a change as they accommodate discrepant experiences (Romberg, 1993).

Tall (1989) identified four types of environments within which students may build and test such schema. Inanimate: where the stimuli come from objects manipulated by the student. Cybernetic: where the system responds to the student according to pre-ordained rules. Interpersonal: where the stimuli come from other people and Personal: within the student themselves. A CAS adds a cybernetic environment to the traditional classroom where students may explore mathematical ideas.

THIS STUDY

The purpose of this study was to monitor students' responses to the use of CAS and to determine if its use did indeed lead to students adopting positive learning strategies including using multiple representations and negotiation of meaning. The research was undertaken in 1998, at the University of Ballarat. This observational study involved a group of 30 undergraduate students who used the computer algebra system, DERIVE 2.55, to assist in both their learning and assessment for an introductory calculus unit

METHOD

The students were taught in two groups, one of 20 and one of 10 students. This grouping was determined by timetable constraints and experienced teachers taught both groups. Students had 4 hours of formal tuition per week with 2 of these hours being held in computer labs. Data was collected throughout the semester from a range of sources: questionnaires, observation and assessed work as detailed in Table 1. These instruments focused on how students used DERIVE, their learning strategies and how they felt about such a use of technology.

Table 1
Summary of Nature and Focus of Data Collected

Nature of Data Collected	Focus of information
Background questionnaire n=28	Previous maths experience Previous technology history Initial response to DERIVE
Lab Classes 10 reports of observations by researcher	Student engagement Student learning methods Student understanding
Solutions submitted by students weeks 1,4,7: total 27	Interesting approaches / errors
Questionnaires: using different questions	Response to DERIVE
Week 3 n=24, Week 5 n=16	Perceived learning
Week 7 n=25, Week 10 n=15	Use of DERIVE
Assignments Marks, n=29	Measure of learning
Representative assignments, n=6	Interesting approaches / errors
Observations by researcher during tests	Use of DERIVE
Test results and scripts	Interesting approaches/ errors
Test 1, n=30, Test 2, n=28	Use of DERIVE
Unit evaluations, n=18	Response to use of DERIVE

Table 2
Percentage of Students using Graphs, Tables or Algebra to deal with Functions

	Never	Occasionally	Half of the time	Very often	Total
Q1 Look at Graph	0	12	46	42	100
Q2 Look at Table	13	58	25	4	100
Q3 Use Algebra	0	43	22	35	100

RESULTS AND DISCUSSION

Students Used Multiple Representations

Class observations, made by the researcher, indicated that all students swapped freely between algebraic and graphical representations of functions but they seldom used tables unless specifically directed to do so.

This observation was confirmed by the questionnaire results. In week 3, students were asked to respond to three statements 'If a problem involves a function I ...look at the graph...look at the table...use algebra'. The results, summarised in Table 2 emphasise that students do make use of multiple representations but show a preference for graphs.

Students' lack of use of tables to solve problems may reflect the use of different representations in the teaching of the course. In classroom examples, done by either teacher or students, tables were almost exclusively used in the developmental phase of conceptual understanding and rarely in problem solving.

In addition it may be that students did not find the tabular representations helpful to their understanding of functions. The use of patterns illustrated in tables was been put forward as a good tool for developing algebraic thinking (Australian Education Council, 1990).

However MacGregor & Stacey, (1992), found that there is a 'tendency for students to focus on the differences between successive values of the dependent variable, rather than on the relationship connecting dependent and independent variables' (p366). In the context of functions this means that many students do not focus on the patterns which could help them establish the function's rule and when looking at limits they may not link the change in the dependent variable to the change in the independent variable.

Students Consulted Each Other as they Negotiated Mathematical Meaning

Students were timetabled for two hours per week in computer laboratories. They had worksheets to guide them and questions from the textbook to complete. There were always some questions that encouraged the students to explore patterns and use inductive logic. Observations made by the researcher, during these computer laboratory classes, recorded that, from the beginning of semester, students worked together and discussed their work. In week 9 the following note was made:

> Most students now work in pairs or threes. Some share a computer while others choose to work side by side. They compare and discuss their results. They compare and discuss what is on their screen and how this applies to the problem. Occasionally the discussion widens with students standing to view screens in rows ahead or rotating the screen for others to see.

The students were also questioned about the discussions that took place in the labs. For example, in week 3, students were asked to respond to statements about their shared computer use. Their responses, summarised in table 3, confirm the observation that most talking taking place in the computer laboratory sessions related to consultations about mathematics rather than their social lives.

Table 3
Percentage of students discussing Mathematics while using CAS

	Never	Occasionally	Half the time	Very often	Always
When we share a computer we discuss what is on the screen	0	9	16	46	27
When we share computers we don't talk about maths	19	52	29	0	0

Since computers were introduced to classrooms back in the early 1980's it has been reported (for example Phillips, 1984) that students exploring mathematical ideas on computer terminals were eager to share their work and learn ideas and strategies from each other. In 1984 students' enthusiasm could have been related to a novelty effect however this study's results concur with Heid & Zbiek (1995) and Thomas, Tyrrell & Bullock (1996) suggesting that while computers are now commonplace they still act as a catalyst to focus discussion.

Students Consulted the Computer as they Negotiated Mathematical Meaning

Students not only used the CAS to find answers to specific set questions but also to their own questions. The CAS was used both as a tool and an independent expert. The tool enabled students to look at a lot of examples quickly and know that these results were correct within the limits of the technology. It also allowed students to look at the same example represented in different ways or to swap between these displays. The independent expert allowed students to test ideas. For most topics, students were observed to effectively use the CAS to change values of coefficients and exponents in order to make and test conjectures. This process was used to establish patterns and develop rules inductively. While students sometimes became frustrated when their conjectures failed they seemed content to keep trying until a correct rule was established.

Unit evaluations confirmed that most students had deliberately interacted with the CAS to extend and clarify their mathematical ideas. Students were asked to respond to statements about their use of DERIVE and these results are summarised in table 4.

Table 4
Percentage of Students Responding Positively to the Use of DERIVE

Questionnaire statement	% positive responses
DERIVE had helped me to see patterns	71
I tried out ideas using DERIVE	70
I used DERIVE to try changing functions to see what happened.	76
I find using DERIVE helps me understand mathematics	65

These questionnaire responses demonstrated that most students responded positively to the use of CAS in a cycle of student input – computer feedback – student reflection – leading to new student input. The 'reflection' stage usually involved group discussion. In this negotiation the students consulted the computer as an impartial expert.

The technology had some limitations. Observation of students in laboratory classes and analysis of solutions to the first problems sets showed that the limitations of the graph plots, on the machines being used, caused some confusion. For example, in graph windows, axes were not labelled, piece-wise defined functions, with discontinuity, appeared with vertical lines joining the 'jump' in the graph and straight lines were sometimes distorted. In the algebra window the package gave complex as well as real solutions to equation. Most of these students had no experience of complex numbers and so did not recognise the symbols. In these situations the computer was not a helpful expert in terms of providing clear simple solutions for the student. However these facets of the CAS prompted important discussions between students and with the teacher. It drew attention to features of the functions that some students may have overlooked. The constraints of the package presented fewer problems later in the semester as the students rapidly gained familiarity with both the concepts and the technology.

Just as interpersonal communication does not always leave all the people involved with the same understanding, reading a computer screen can result in different perceptions by different users. Each student's response was influenced by their previous experience of algebra, graphing and their visual perception. Tufte (1990), noted that there may be visual perception problems associated with figures which have several lines close together. Analysis of students' worked solutions to questions in week 1 showed an example of this. Students were asked to sketch, on one set of axes, x^n for increasing powers of n. A number of students drew graphs that doubled back or even became 'horseshoe' shaped. These students believed that they had carefully copied what they saw on the screen.. When the background surveys of the students who made these errors were checked it was found that they had a limited school mathematics background in algebra and calculus.

A key difference between using a computer package and working with pen and paper is that the CAS responds to the student's input, thus providing feedback for the ideas the student is exploring. In some senses negotiation takes place as the student works with the CAS. A student, for example, may make a conjecture about the salient features of a particular gradient function. This idea may be tested immediately by finding the derivative algebraically, and then graphing. The student quickly refines their understanding as they recognise examples that do and do not belong to the pattern they are seeking. A cycle of 'student input → computer feedback → student reflection → new student input' repeats until the student feels that they have successfully understood the concept involved. In this way consultation with the impartial computer aids students as they build and refine their understanding of concepts.

Students' Preferred Learning Strategies

As part of the Week 5 questionnaire students were asked to identify the tools which they would select for speed, confidence, and understanding in response to a range of mathematical questions. The most common choices of the 16 students who responded are summarised in table 5. For each category, the table indicates the tool chosen by the highest percentage of students and the percentage making that choice.

Table 5
The Tool Most Frequently Selected for Speed, Confidence and Learning

For this type of problem	Speed	Confidence	Learning
1. Solve for x : x 2x + 1 = 5	PP (75%)	PP (75%)	PP (100%)
2. Solve for x: (x-4)(x+1)=0	DA (56%)	PP (50%)	PP (83%)
3. Find the intercepts and shape of a y=2x+1	DG (56%)	DG (56%)	PP (63%)
4. Find the x and y intercepts for $y=x^2+5x+6$	DG (38%)	PP (47%)	PP (83%)
5. Find the turning point for $y=x^2+5x+6$	DG (47%)	DG (47%)	PP (44%)

Legend: PP (pen/paper) DA (Derive-Algebra) DG (Derive Graph) CC (Calculator- computation) CG (calculator- graphs) OT (Other –please specify)

These results indicate that while students often felt that using CAS was fastest, and they might be more confident of obtaining a correct result with DERIVE, they commonly felt that they learnt most when they worked by hand with pen and paper.

This result was reinforced in the unit evaluations when students were asked if they felt that 'CAS might offer fresh hope to students who had previously experienced difficulty with algebra'.

> Student A: The use of a CAS would, in some cases, give fresh hope but only if there is still a grounding in the concepts behind the problem.
>
> Student B: I don't think it helped my understanding, because while the computer does the work for you it is too easy not to fully understand what is going on.
>
> Students C: CAS offers the student the answers without really needing the basics. I personally would have enjoyed more time spent on the basics.
> Student D: It does assist, however a basic understanding of the maths is required first. You can not expect the computer to do everything for you. Personally I would rather do theoretical algebra and calculus, it enables me to understand it better and I have more confidence.
>
> Student E: It is a good idea, however although the use of computers will give you the correct answer (as long as entered correctly) it does not let the user know what they have done or how they would go about getting the answer if they were to do it manually.
>
> Student F: CAS in this unit has been a great tool I have learnt to use. I have found it very useful and it is a great piece of knowledge I could use in later life.

This sample is representative of the responses. It indicates that most students felt that the use of CAS contributed to their learning but that pen and paper experience was more important. This result is not surprising as the previous experience of these students in school mathematics focused on pen and paper routines. This was the skill valued by school assessment and therefore highly valued by the students. Background information collected showed that only 11% had used a graphical calculator at school, 37% had used a computer for some maths related purpose and 11% had used both a graphical calculator and a computer at school. Using technology in learning and assessing mathematics was a new experience for most of this group and so many were unsure whether this was legitimate mathematics

CONCLUSIONS

This paper has discussed three positive learning strategies that have been adopted by students using CAS and the their response to this use of technology. The results confirm past findings about the way students learn mathematics and also affirm the place of CAS as a facilitator of positive learning strategies.

Students frequently made use the CAS to explore mathematical ideas by swapping between representations. They showed a preference for graphs and algebra while making little use of tables. Future refinement of both software interface and teaching materials may extend the use of multiple representations.

Students working with CAS discussed their findings with each other and treated the CAS as an independent expert. These interpersonal and cybernetic stimuli helped to extend their mathematical thinking and modify their mathematical schema. Sometimes the limitations of the technology resulted in to confusing responses. Again future development in both technology and teaching materials could solve these problems.

Students expressed reservations about using CAS to learn mathematics. They expressed a view that real mathematics was that done with pen and paper. The availability of CAS raises the question of what mathematics should be taught and assessed at school and undergraduate level. This challenge has been recognised for nearly two decades but it is yet to be resolved.

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