Modes of Representation in Students' Explanations

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Mathematics curricula commonly promote the development of problem solving, reasoning and communication skills. Students need to be able to communicate their mathematical ideas; written communication is therefore an important part of the assessment process in mathematics. When writing explanations students need to consider and choose an appropriate mode of representation for their response. This paper examines students' modes of representation used in solutions for two problem solving tasks and considers the issue of judging mathematical communication.

Communication is an integral feature of current curriculum reforms in mathematics (Australian Educational Council, 1991; Department for Education, 1995; Ministry of Education, 1992; National Council of Teachers of Mathematics, 1998). Communication is a key aspect of learning. Students need to communicate if they are going to make sense of mathematics; communication has both cognitive and social significance in the classroom (Hiebert, 1992). It is through communicating mathematical ideas that students become engaged in 'doing' mathematics. "Doing mathematics means agreeing on assumptions, making assertions about relationships, and checking if the assertions are reasonable" (Hiebert, 1992, p. 444).

> Communication in and about mathematics serves many functions. It helps to (1) enhance understanding, (2) establish some shared understandings, (3) empower students as learners, (4) promote a comfortable learning environment and (5) assist the teacher in gaining insight into the students' thinking so as to guide the direction of instruction. (Mumme & Shepherd, 1990, p. 18)

It is asserted that the use of assessments in mathematics should be consistent with the goals of educational reform and reflect current curriculum statements (Begg, 1998). These goals reflect a change in emphasis from traditional content to the processes. Assessment tasks should therefore measure achievements in the mathematical process skills such as problem solving and communication. They should be carefully constructed so that students are encouraged and motivated to solve problems and communicate their mathematical thinking (Clarke, Waywood, & Stephens, 1993; Miller, 1991; Szetela & Nicol, 1992). An appropriate task should also provide quality insights into students' mathematical understanding National assessments in New Zealand in recent years have shown a trend to provide students with more authentic assessment tasks and to have students write explanations and justifications as part of the solution process (Bicknell, 1998). Questions are more likely to be constructed so that they allow students to construct solutions and provide visible records of their thinking processes, instead of simply recording an answer.

Problems that require students to give explanations and justifications are usually cognitively more complex. "Cognitively complex tasks should be challenging and unusual so that students have to construct strategies and procedures for solving the problems; rote methods should not suffice" (Magone, Cai, Silver, & Wang, 1994, p. 320). The tasks should also allow multiple forms of representation, multiple solution methods or even multiple answers. However, Linn, Baker and Dunbar (1991) warn that it cannot be assumed that open-ended tasks require more complex cognitive processes than response-choice items. An analysis of a task's complexity can be based on the task features that make it complex and a consideration of actual student responses (Magone, Cai, Silver & Wang, 1994).

Recent developments in cognitive science highlight the value of examining more than the correctness of answers to problems. It is important to also examine characteristics of the solution including problem solving strategies, errors and representations (Resnick, 1988). Representational tools are a valuable aid to the thinking and communication processes. The National Council of Teachers of Mathematics (1998) identifies representations as one of its standards and states that: "Mathematics instructional programs should emphasize mathematical representations to foster understanding of mathematics"(p. 204). Students may use multiple representations but it is the act of representing the problem that helps students to articulate, and clarify ideas; the ideas can then be evaluated and extended.

Allowing students to choose modes of representation makes it more likely that students will give an indication of their problem-solving processes; it helps them to show their solution strategies (Magone, Cai, Silver, & Wang, 1994). An examination of modes of representation is a valuable part of the process of describing student performance qualitatively (Cai, Magone, Wang, & Lane, 1996). Explanations and justifications can be reliably and efficiently judged, providing diagnostic information on the level and quality of individual students' understandings (Niemi, 1996). The information from the analysis of students' responses can then be used in classroom instruction so that together the teacher and students develop a qualitative framework for interpreting and evaluating explanations

The Study

The findings that follow are taken from a larger study which focused on the writing of explanations and justifications. The research was conducted at a large provincial co-educational secondary school and involved 36 students from two classes. The students completed a problem-solving task sheet of five problems. These items were modelled on externally-mandated national examination questions from the New Zealand School Certificate examination in mathematics. The first part of the examination uses response-choice items and short answers. The second section has more cognitively complex tasks.

The problems presented to students were, in contrast to short-answer and multiple choice items, 'open-ended'. Open-ended tasks according to Cai, Lane and Jakabcsin (1996) are more informative and make mathematical communication part of the assessment. They require students to not only produce answers but to show solution processes and give justifications for answers. In the tasks provided, students were required to not only select and produce an answer but to show their working; to explain and justify their answers. Follow-up semi-structured interviews were conducted to find out more information about the student's responses. The students were able to refer to their solutions during the interview.

Two problems from the study are presented and a qualitative analysis of student responses provided. Students' responses were analysed for the quality of the mathematical argument and the mathematical correctness; this has been reported elsewhere (Bicknell, 1998). This paper focuses specifically on the modes of representation used for the explanations.

Results and Discussion

Problem 1

The first problem (Figure 1) was a measurement problem designed to assess students' understanding of volume and their ability to choose and justify selecting a particular thickness (ie the concept of average). They were also required to provide an explanation and finally give a justification for the choice of degree of accuracy.

Problem 1:

Derryn investigated the packaging of snack bars. She measured the packet with a ruler and found that it was 16.5 cm long, 9.2cm high, and 4.6 cm wide. She calculated the volume to be 698 cm^3 (3sf).
The packet had 6 muesli bars in it.
Each muesli bar was 7.5 cm long, 4 cm wide but the thickness of the bars varied between 2.6 cm and 2.8 cm.
Find the volume of the 6 muesli bars as a percentage of the volume of the packet.

- *Explain what you are calculating at each step and show your working.*
- *Round your answer appropriately, stating the degree of accuracy*
- *Justify why you have chosen this degree of accuracy.*

Figure 1. The measurement problem.

Responses were analysed and categorised according to the mode of representation used. The students' modes of written representations were categorised as: symbols only; symbols and words; or a combination of symbols, words and diagrams. Only three of the students provided a response using symbols only, devoid of any verbal explanation of what they were calculating. The remainder all included verbal explanations with 56% of the students writing an explanation using symbols and words. A combination of symbols, words and diagrams was the mode of representation chosen by 36% of the students.

Kim's solution was an explanation using symbols only (Figure 2). She lacked confidence in her ability to give a satisfactory explanation and was unsure as to whether her chosen mode of representation was sufficient. In the follow-up interview Kim reflects on her answer: "I'm not quite sure whether I've explained it. It's a bit messy and I'm missing the labels so people won't know what I'm doing. I don't explain at the start what I'm doing."

$7.5 \times 4 \times 2.6 = 78$

$7.5 \times 4 \times 2.8 = \underline{+84}$

(average) $162 \div 2 = 81 \times 6$

$\frac{486}{698} = 69.6\%$ is muesli.

Figure 2. An explanation using symbols.

Jessica's answer (part of which is shown in Figure 3) is an example of a comprehensively written explanation using symbols and words. When interviewed Jessica revealed that she was uncertain about whether this mode of representation was what was expected and was concerned that she had written too much. She recognised that she invariably wrote excessively and liked to write in prose so that an examiner was clear about what she had to say.

Firstly, the volume of the 6 muesli bars have to be found. As the thickness of the bars varied between 2.6cm and 2.8cm the average thickness has to be found. To do this you minus the smaller thickness from the larger thickness and divide by two, then add the smaller thickness as seen below

$$\frac{\text{larger thickness} - \text{smaller thickness}}{2} + \text{smaller thickness} = \text{average thickness}$$

$$\frac{2.8\text{cm} - 2.6\text{cm}}{2} + 2.6\text{cm} = \text{average thickness}$$

$$\frac{0.2}{2} + 2.6 = 2.7\text{cm}$$

Now that an average thickness has been found the volume of one muesli bar can be found, then multiplied by six to get the total volume of the 6 muesli bars

volume of 1 muesli bar = length x width x thickness
= 7.5cm x 4cm x 2.7cm
= 81 cm^3

To find the volume of the 6 muesli bars you now multiply the answer from the volume of 1 muesli bar by 6

Volume of 6 muesli bars = volume of 1 muesli bar x 6

Figure 3. An explanation using symbols and words.

Many students included a diagram in their responses. Jan's is one example (Figure 4). Jan justified her use of a diagram and commented: "I like using diagrams; it helps you know which bit you're talking about." Another student explained: "A scale-type drawing gives you a better picture of what you're doing. It shows you how to find things and if you are thinking appropriately."

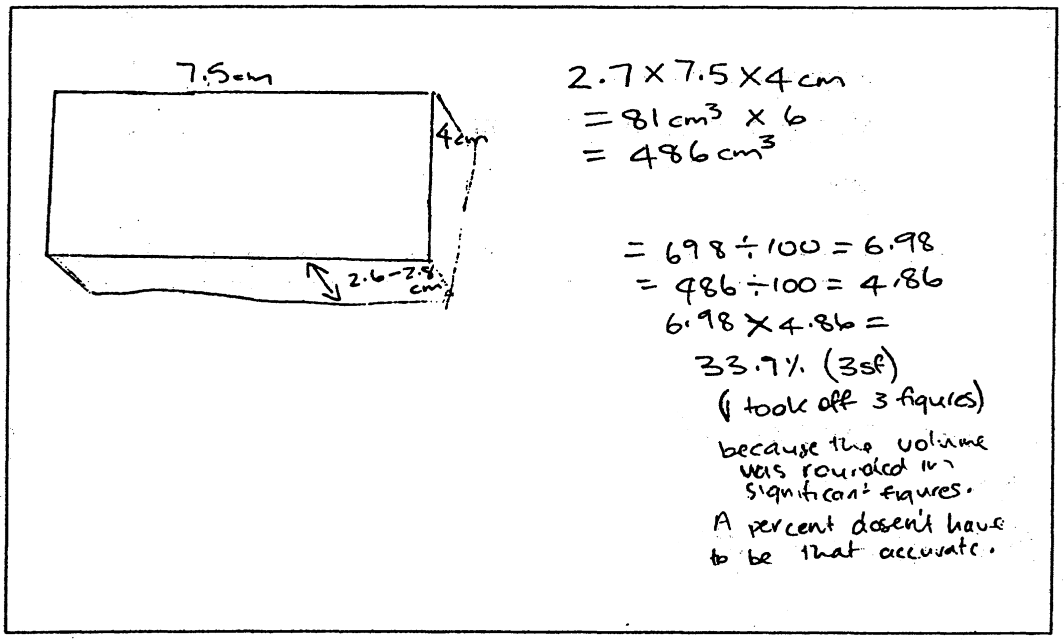

Figure 4. An explanation that includes a diagram.

There were common concerns from the students about not knowing whether they had provided a satisfactory explanation and whether their chosen mode of representation was what was required. They really had little confidence that a teacher would accept their mode of representation as appropriate. Those who combined modes were "covering their options" and ensuring that their explanation was comprehensive enough.

Problem 2

In order to answer this problem (Figure 5) the students had to firstly extract the two items of data needed and in their response convey the idea of quantity. Secondly, they had to give a process description explaining the calculation. The majority of students (89%) were clearly able to state the two items of data. The difficulty of defining and interpreting the scope of the explanation was illustrated by the number of students (7) who elaborated on how the data was to be collected. These ideas related to how to conduct a survey, data gathering techniques, and finding averages.

> Problem 2
>
> James and Richard are gathering data for a survey on students' lunchtime eating habits. They wish to find the percentage of students who buy their lunch from the school canteen.
>
> State the two items of data they need to gather and **explain** what calculation they need to make in order to be able to calculate the percentage of students who buy their lunch from the school canteen.

Figure 5. The statistics problem.

The responses were categorised according to the mode of representation, namely; the use of words, symbols, algebraic expressions and whether a worked example was used. He resulting distribution is shown in Table 1.

Table 1
Modes of Representation for Problem 2

Mode of Representation	Frequency
Explanation using words only	11
Explanation using words and symbols	15
Explanation using an algebraic statement	2
Explanation using words and an algebraic statement	2
Use of a worked example	1
No explanation	5

> Divide the number of people who buy their lunch buy the total number of students and multiply by a hundred to give you a percentage of people who buy their lunch.

Figure 6. A verbal explanation.

Many of the students (30%) provided a verbal explanation as shown in Richard's example (Figure 6). Richard explains why he wrote his explanation exclusively in words: "I thought that I'd get more marks writing it down like that, but again I probably didn't really know. I saw it as a 50% chance of getting it right."

The preferred mode of representation was to combine both words and symbols (Figure 7). Paula explains: "I wrote it all down and then replaced what I'd said with an equation. I think that in order to make sure that you've got what the marker requires you need to write it all down."

The two items of data James and Richard need are
① the number of students who buy their lunch from the school canteen
AND
② the total number of students in the survey

Once they have collected their data they need to divide the number of students who buy their lunch from the school canteen by the total number of students in the survey then multiply that answer by 100

Percentage of students who buy their lunch from the school canteen = $\frac{\text{no. of students who buy lunch from school canteen}}{\text{Total no. of students surveyed}} \times 100$

Figure 7. An explanation combining words and symbols.

Two students used only algebraic statements but explained the meaning of each variable, whilst five students did not attempt to explain the required calculation. A student who provided a comprehensive verbal and algebraic answer expressed doubt about whether this was appropriate because it all seemed a bit simple. In contrast, Mark believed quite confidently that his solution (Figure 8) was appropriate. He explained: "I answered the question because it didn't say solve it, it just says explain your calculations and I've done that and I've written it in sentence form."

They need to know the total number of people at the school and then find out how many student buy from the school canteen.
Then they will have to find the perctage of students buying lunch ~~[illegible]~~ and percentage of students not buying from canteen.

Figure 8. An incomplete verbal explanation.

Conclusions and Implications

Students represented solutions in a variety of ways; they used symbols only, words only or used a combination of symbols, words, and diagrams. The students chose varying forms of representation for each of the problems for a variety of reasons. Some students felt that one form over another would receive more marks and was therefore superior in assessment terms; others chose a form that they felt was sufficient to explain their problem-solving processes. A few students duplicated their explanations using different modes to ensure that they had provided what was expected; to ensure that they were covering all 'expected' options and therefore would not sacrifice any marks in the assessment process.

Students had personal preferences for particular modes of representation in their explanations. One of the concerns is that this preference was not based on a sound mathematical reason. There was a clear common concern about not knowing what was the expected mode of representation and the question surfaced: "Is one mode of representation superior to another?" They expressed doubts about whether in the assessment process their chosen mode of representation would gain full recognition. This element of insecurity and doubt about what is the best or preferred way for representing solutions has direct implications for teachers.

One of the teacher's goals in the mathematics classroom is to promote the development of communication skill. The teacher's role is to help students learn to use representations flexibly and appropriately in their explanations and justifications. Students need to develop a repertoire of representations so that they can choose an appropriate mode to support a particular solution. This goal can be achieved through presenting students with different levels and modes of representation and asking students to evaluate responses in terms of the quality of the mathematical communication.

The classroom situation should also provide students with the opportunity to share and discuss their alternative modes of representation. This process contributes to the development of sociomathematical norms (Yackel & Cobb, 1996). Teachers can help establish a normative understanding of what is an acceptable explanation and justification. Criteria can then be established for judging mathematical communications and for establishing high quality explanations using a variety of modes of representation.

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