The Understanding and Use of Trigonometric/Algebraic Knowledge during Problem Solving

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An important feature of students' understanding of algebra and trigonometry is the relations and representations that are constructed between these two areas of school mathematics. In this study we report one students' understanding of the concepts before and after two lessons. The student developed knowledge about the topics that appears to be consistent with what the teacher was aiming at. However, in a problem-solving situation, the student activated knowledge that was different and incorrect. The results provide tentative support for the argument that students sometimes develop 'garbled knowledge'.

Introduction

There is substantial agreement that an important goal of mathematics teaching is to develop students' understanding of concepts and procedures associated with concepts. An important element of such understandings would be the establishment of deep links among concepts, facts and procedures (Hiebert & Carpenter, 1992). The need to examine connections that students construct has also been endorsed by major curriculum reform documents (National Council of Teachers of Mathematics, 1991). A number of studies have explored the type of understandings developed by students, and the quality of teaching provided by teachers. While these studies have generated a useful body of data, in the main, few have examined the student's understanding in relation to what has been presented by the teacher in the classroom and the quality of that understanding as revealed in problem solving. This is a report of a study which was aimed at providing a detailed analysis of one student's understanding of trigonometric knowledge after a series of two lessons.

There are many things to examine in a student's understanding of mathematics. Carpenter and Lehrer (1999) argue that different types of mental activity contribute to the growth in understanding: the relationships or connections made during knowledge construction, the application of knowledge, reflection about this knowledge use, and articulation of what is known. In this report we set out to seek evidence of each of these types of activity in a student whose lessons on gradient we had observed and analysed. Our involvement in this student's class gave us the opportunity to observe what Sierpinska (1994) note to be three key aspects to mathematics understanding: what is being understood, the context in which it is understood and the operations activated during the process of understanding. The 'what is being understood' could include concepts, facts, principles and procedures. An equally important component of this understanding is the relationships that students construct among the different components of their knowledge in the context of the classroom activities designed by the teacher. As the students' attain higher and more sophisticated levels of understanding, one could expect students to show evidence of more varied and extended schema of relationships that could be used to solve problems of increasing complexity. It is in the course of this problem solving that the quality of the knowledge schema is tested.

Teaching and the Construction of Connections

The above framework suggests that in order to improve mathematical understanding, classroom teaching needs to focus on helping students construct schemas that are rich in content and relations. A schema-based analysis of knowledge constructed by students implies that the quality of explanations employed by the teacher and the range of examples and metaphors used by teacher could be expected to have a major influence on the nature of schemas developed by students. This line of reasoning suggests that we could generate important data about students' understanding by examining the relationship between schemas that teachers activate during their teaching and those constructed by the students. In this project we examined this relationship by documenting in detail the knowledge schemas activated by a teacher during teaching about gradient and comparing one student's schema before and after the instruction and the use of that schema in a problem solving activity.

Method

Participants

Participants in this study were part of a larger study in which a group of experienced teachers who were required to plan and teach a series of two lessons to their Year 10 classes. The teachers came from a pool of participants within the states of Queensland and Western Australia. They were assessed to be exemplary by the Education Departments and Mathematical Associations. In this report we consider one teacher and a student from his class. The male high school teacher had twenty years experience in teaching of mathematics at a senior secondary level. He was also the head of the mathematics department of his school.

The student was considered to be an average student on the basis of his performance in a series of four tests and comments from the teacher. On the basis of classroom tests the teacher rated him as an average student in a class that was preparing students to take mathematics as a major subject in the final two years of schooling. The teacher and student will be referred to as Gary and Tom respectively in this paper.

Tasks and Procedure

The purpose of the study required that we assess the knowledge of the student before and after instruction by the teacher. Three tasks were developed for the purposes of assessing students' knowledge about geometry. The first task, Free Recall (FRT), required students to talk aloud about any idea that they could associate with the topic of plane geometry and about shapes and their properties. The second task, the Problem Solving task (PST), consisted of four problems. All the four problems were related either to plane and/or coordinate geometry. The first problem involved the use of Pythagoras's theorem in a rectangular coordinate system. Solution of the second and third problems required an understanding of the relationship between properties of squares and rectangles, segmentation of their areas, and knowledge of the gradient of straight lines in a rectangular coordinate system. The fourth problem, again necessitated an integration of knowledge of properties of square and right-angled triangle.

The final task, Geometry Probing (GPT), was designed to obtain further information about student's talked about during the Free Recall task. In the GPT the student was required to respond to a number of questions each of which aimed at probing what was said during the first session. The probes were designed to provide the student with opportunities to display his available knowledge of features of concepts relevant to this area of mathematics. The use of the above techniques has been argued to provide rich data about interconnections that exists between knowledge units, and their structure (Royer, Cisero, & Carlo, 1993).

The student met with us on three occasions and undertook one of the tasks in each session, in the order indicated above. In the first session he was invited show his knowledge of geometry by talking, drawing and writing in response to the FRT. During the second session he was asked to complete the PST which involved generating solutions to four problems. He was encouraged to talk aloud during the solution attempts. In the third session we sought the student's responses to the probe questions. All three sessions were video-taped and transcribed for subsequent analysis. Data from the above three tasks were used in the development of pre-teaching concept map for the student.

Gary made a decision to spread his introductory lessons on linear equations across two lessons. Both his lessons were video-taped and later analysed. Part of the analysis involved identification of the concepts, principles and procedures taught during the lessons and it is this that we will report on here. The components of the lesson that addressed our focus concept gradient and associated concepts are shown in the teacher's concept map (Figure 1). In other parts of the analysis we rated the lesson in terms of moves made by the teacher to establish and strengthen connections in the knowledge schema that were the focus of the lessons (Lee, 1998). For the purposes of the present study we focussed on Gary's discussion of the concept of 'gradient'. In this regard Gary's lessons allocated substantial amounts of time to both schema establishment and schema strengthening.

Following the lessons, the student was asked to participate in two further interviews. During the first of these post-teaching interviews, the student attempted to solve two problems. We asked him to 'think aloud' as he worked on these problems. The problems were based on the content discussed during the class lessons and were designed to elicit responses from the student about his understanding of gradient. During the second interview the student

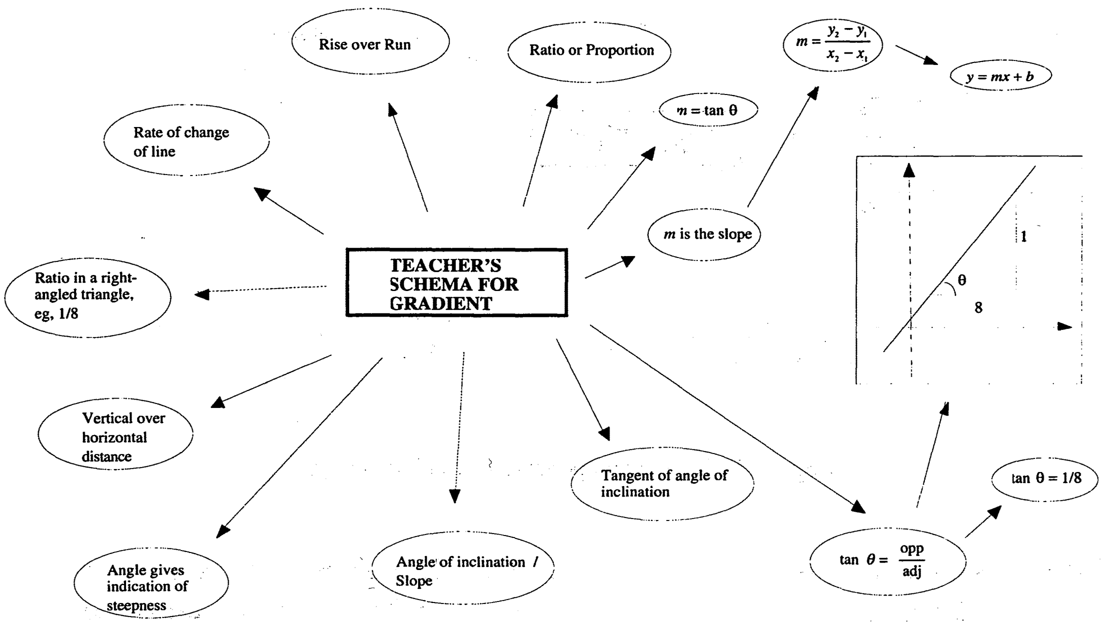


Figure 1: Components of teacher's lesson.

responded to a series of focus questions. These focus questions were used to gather information about student's perceptions about what the teacher was aiming at and developments in his understanding of concept of gradient. In these interviews we gave Tom the opportunity to articulate and reflect on his understanding of the focus concepts.

Data from the pre- and post-teaching interviews formed the basis for the development of concept maps for the student. These concept maps were anchored by a number of subsidiary concepts for gradient. The subsidiary concepts included students' knowledge about right-angled triangles, the coordinate system and properties of lines and parallel lines. For the purpose of this paper we have included concepts maps for right-angled triangles (Figure 2) and the coordinate system (Figure 3). In both these figures, concepts that appear in oval-shaped (not shaded) figures were activated prior to and after teaching. Concepts that appear in rectangle-shaped figures (not shaded) were activated prior to teaching only. Concepts that appear in rectangle-shaped figures and shaded were only activated after teaching.

Results

Figure 1 shows that teacher has activated a range of ideas during his lessons. Analysis of the sequence of his presentation indicated that Gary spent considerable time discussing the idea of gradient informally before formalising it in terms of proportion/ratio, tangent of an angle and coordinate geometry. This informal discussion drew explicit links between gradient and geographical features near the school. Gary's representation of gradient addressed included the fundamental notion of slope and its relation to the concept of function and it is clear in Figure 1 that there was an amount of redundancy associated with representation of the concept of gradient. The state of his knowledge network is rich and extensive which Stump (1999) argued to be an important element of a better of understanding.

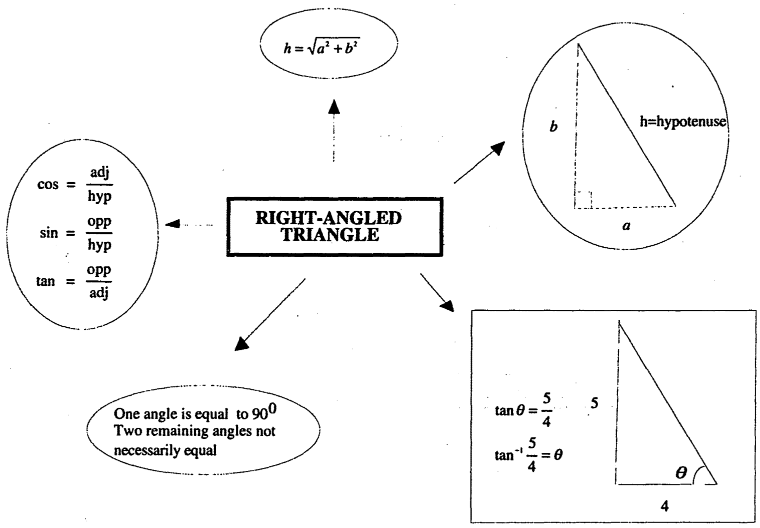


Figure 2. Concept map for right-angled triangles.

Figure 2 shows that prior to the lessons Tom had already built up a number of concepts about the properties of right-angled triangles including those related to gradient. This is indicated by concepts inside the unshaded oval shapes. The bottom right-hand corner shows that Tom appear to have acquired something new from the lessons about the relationship between gradient and tangent of angle. From Figure 3 we see that Tom's understanding of coordinates were reasonably complete even before the lesson. Taken together, Tom appears to have developed a level of understanding of gradient that might be expected to be useful in the problem solving that would follow. For example, on the basis of this understanding we would expect him to be able to calculate the gradient and equation of a line joining two points in a Cartesian system. We tested our expectations about his understanding in our post-teaching interviews.

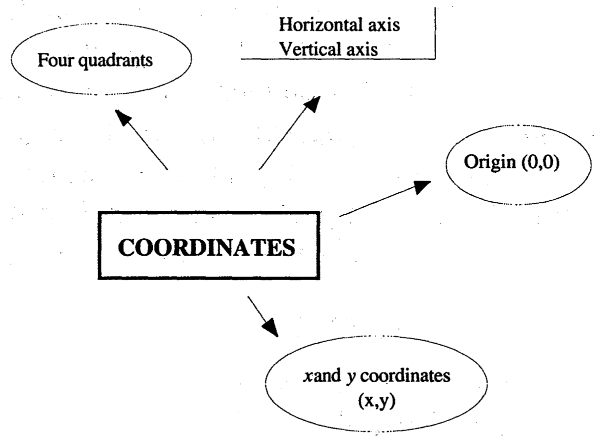
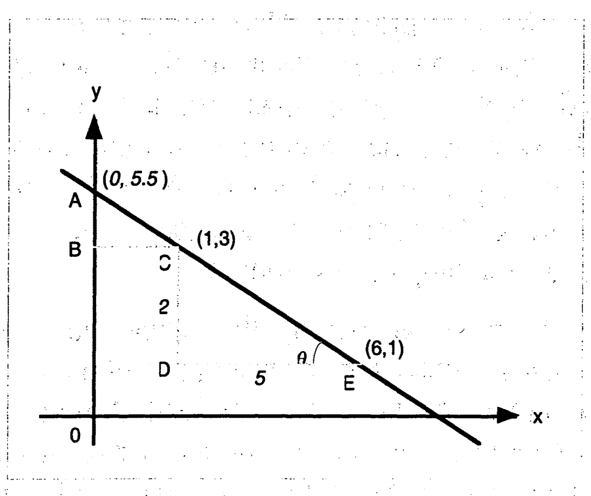


Figure 3. Concept map for coordinates.

Figure 4 shows Tom's attempt at completing the problem which required him to find the y-intercept and equation of the line ACE. Analysis of Tom's think aloud activity and transcripts showed that he followed the seven major steps set out below Figure 4. In Step 1, Tom found the slope by using the notion of vertical distance over the horizontal distance within the right-angled triangle CDE. The distances were found by looking for differences in the y and x-coordinates of points C and E. He repeated this in Step 2 but reversed the order of the coordinates. Both procedures yielded the same result, though Tom did not note this. He then found the tangent of the angle θ in Step 3. In Steps 4 and 5 he incorrectly



1. $\frac{3-1}{1-6} = \frac{2}{-5} = -0.4$
2. $\frac{1-3}{6-1} = \frac{-2}{5}$
3. $\tan\theta = \frac{2}{5} = 0.4$
4. Inv tan .4= slope
5. $\tan^{-1} .4 = m$
6. $y = mx + b$
7. $y = .4x + 5.5$

Figure 4. Problem and Tom's solution attempt.

concluded that 'inv tan 0.4' is equal to the slope of the line. This is a new element in his currently activated schema. It was not mentioned in the pre-teaching interview and was not discussed in the lessons. We might reasonably expect it to be connection that he had madepreviously in learning about inverses of trigonometric ratios. In Step 6, Tom activated the general form of the equation for a straight line. His actions in Step 7 show that he understood the various components of the general equation and replaced the values for gradient (*m*) and y-intercept. The value of b is generated via his somewhat garbled reasoning that 'for every one across you go 2.5 up; so AO (the y-intercept) equals 3+2.5'.

Tom could identify the appropriate coordinates (steps 1 and 2), could establish the tangent of the angle (step 3), and could recall the correct symbolic representation of the linear equation (step 6). However, his problem-solving moves show that he was unable to set up an appropriate set of links between gradient, the coordinate system, tangent of an angle and the representations of the concept of gradient presented in the lessons. His schema for ratio/proportion used to calculate gradient was different from what the teacher attempted to establish as 'rise over run' in the lessons (Step 7). In this move he showed that he had difficulty interpreting the relationship captured by the expression $m = \frac{y_2 - y_1}{x_2 - x_1}$, despite the fact that he had earlier used part of this knowledge in steps 1 and 2. Thus at the end of this problem-solving attempt Tom shows clear evidence of confusion about the relationship between m, tan θ, and slope of the line.

Discussion

The expectations of problem-solving performance we had built up on the basis of our analysis of the student's pre-teaching interview and of our analysis of the lessons were not confirmed. Instead we were left with strong evidence that a garbled knowledge schema had been established by this student after two class lessons. What do we want to make of this?

First we do not want to make a final judgement about the quality of the teaching, which we earlier rated as good, or a final judgement about the student's knowledge state. We have a very restricted sample of both this teacher's teaching and this student's knowledge, a sample that would not support such judgements.

Yet consider what information this episode does provide. Here we attempted to briefly open a window on this student's schema knowledge. Prior to the teaching it appeared to be reasonably robust when we used one form of assessment. But it failed in the test of application. Some of the required links between knowledge components appeared to be incomplete or inappropriate. What would be expected (see Moscovitch, 1996) to be useful context provided by the teacher in situating the concept of gradient in the local geographical context was not reflected in the understanding of this student.

There is much in this student's schema to be sorted out by the teacher in future lessons. However, as teachers we would probably expect the 'average student in a top class' to have done much of this sorting out for himself as the lesson unfolded. A revisiting of the lessons did show that there was no provision in the lesson for the teacher to inspect the student's unguided understanding of what had been shown on the blackboard and in the computer simulations. The examples in the lesson had also included positive slopes.

It may have been that the student's analysis of lesson material was too superficial and that he was 'captured' by the process of calculation (Verschaffel & DeCorte (1997). Indeed there is no evidence in his post-lesson articulation of his understanding of use of synonyms such as 'steepness' for gradient, nor is there evidence of explicit reflection on the state of his understanding. Perhaps the student needs to use these types of activity more actively. At the end of this episode it is clear that the student needed to retreat rather than advance in his construction of knowledge about gradient. If he did not it is unlikely that he would construct a suitably powerful schema. If Tom is like many 'average' students we must consider whether we can allocate them enough attention and time to build such structures, perhaps they need to spend more time covering less mathematics.

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