Function Representations and Technology-Enhanced Teaching

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In recent years calculators and computers have been used more often in mathematics teaching. While their potential advantages in terms of dynamic multiple representations has been recognised for some time, it has not always been clear how to integrate the technology into conceptually based teaching. While it is generally agreed that technology could play an influential role in enhancing mathematical understanding, there is little data about how students' cognition and knowledge building are affected by the instructional use of such tools. In particular, a theoretical perspective on how to teach in such a way as to encourage students to build inter-representational conceptual structures has been lacking. In this study we take up this issue by examining how two experienced teachers plan and incorporate calculator and computers in their teaching. The results suggest that these teachers aim to build knowledge schemas that are rich in connections and representations.

Background

It seems inevitable that not only the nature of the processes, objects and concepts of mathematics, but also the way teachers and students perceive them, will affect the way in which mathematical ideas are taught. These differences often may not make much impact but may be more important when the concept is one which is fundamental to mathematics, such as that of function. Vollrath (1994, p. 70) describes how "*global strategies* are needed for *leading concepts* that permeate the whole *curriculum*, for example the concept of function is a candidate for such a leading concept." What are some of the variables which influence the learning of this leading concept, and how can technology assist teachers in helping students to develop an understanding of it?

Functions can be seen in a number of different ways by students and teachers, and these have been described in terms of different theoretical perspectives. For example, they can be seen purely as an input–output process, or as an encapsulated object (Tall & Thomas, 1991; Cottrill *et al.,* 1996), or some stage in between. In contrast, Hitt (1998, p.125) describes five levels of understanding of the concept of function, each of which is related to the representation of a function, with the highest level being "Coherent articulation of different systems of representation in the solution of a problem." There are a number of different uses of the word representation in the literature (von Glasersfeld, 1987) and we shall follow the definition of Kaput (1987, p. 23) who proposed, following Palmer (1977), that "any concept of representation must involve two related but functionally separate entities. We call one entity the representing world and the other the represented world." This idea applied to the representation of function is particularly important, since functions can be represented in a wide variety of different ways, such as in algebraic symbol form, set diagrams, ordered pairs, graphs and tables of values, and so on.

Translating between these various forms is something which an experienced teacher might take for granted, but to do so one needs to have an overview of the way the definition of function relates to each representation, and how sub-concepts, such as independent and dependent variables, one-to-one, roots, discrete, continuous, etc. are manifest in each representation (Chinnappan & Thomas, 1999; Hong, Thomas, & Kwon, in press). A precise view of these, and the processes which apply to each representation can help one make judgements about processes which, unlike concepts, may not travel well between representations. For example, the process in the graphical domain corresponding to that of solving linear equations by adding multiples of the variables to both sides takes on a completely different form, involving rotations of straight lines.

In our view calculators and computers can assist in the building of links between the sub-concepts in each representation. This is not only because they intrinsically employ multiple representations (Kaput, 1992) or that they can be "dynamic interactive and recording representational media" (Goldin & Kaput, 1996, p. 411). The key property they possess is the fact that the representations can be dynamically linked, with the technology providing fast feedback on student input (Kaput, 1992), so that a change in one representation, such as a table, is immediately mirrored by a corresponding change in the graphical representation. It is true that computers and calculators may appear to present some cognitive obstacles to an understanding of function, such as the limitation of having only discrete variables in a table of values. However, in practice the zoom feature of most graphic calculators admits a variable table range which makes the function domain *potentially* infinite. Figure 1 shows the potential of technology to provide a central role in the linking of four key representations of function.

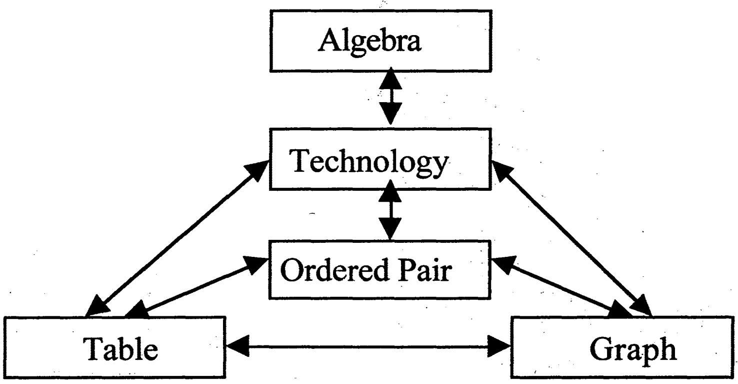


Figure 1. Technology as a catalyst in the linking of function representations

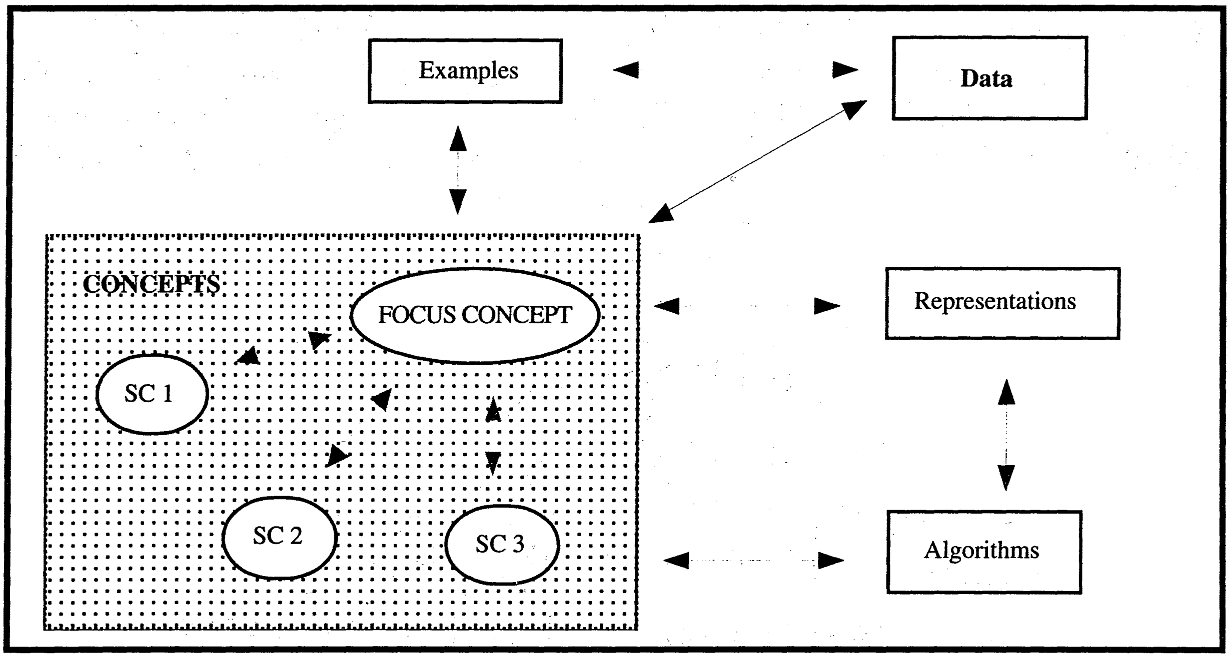
Previous research (Chinnappan & Thomas, 1999) has investigated the pedagogical value of teachers' rich conceptual function schemas, and Vollrath (1994) has suggested that mathematics teachers need to have these if they are to be able to teach function on the basis of conceptual relationships. Research by Williams (1998) shows that it is not always the case that students have such well developed conceptual schemas for function, and Even (1998, p. 119) concluded that the prospective secondary teachers in her study "had difficulties when they needed to flexibly link different representations of functions...". In addition, teachers' schemas have to include an understanding of the role of technology in facilitating inter-representational links in order to be able to use it effectively. This paper describes the kind of pedagogical schemas (Shulman, 1986) which two teachers have developed, focussing on the way they help them think across the representations, and how this has proven beneficial in their teaching.

Method

This research employed a case study methodology, considering the function schemas of three experienced teachers, A, B and C, with 17, 16 and 31 years teaching respectively, and each of whom had also made extensive use of calculators and computers in their teaching. The teachers were given a free response interview where they were encouraged to talk about 'functions' and how they would teach them. The interviews were recorded on audiotape and later transcribed for analysis. In addition, four lessons of teachers A and C were observed and recorded on video. The interview and lessons of teacher C, Margot, have been described elsewhere (Chinnappan & Thomas, 1999), and this paper discusses some of the data from teachers A and B.

Results

In our analysis of the data we were interested in identifying teachers' knowledge about functions and the use of technology in fostering such knowledge. Figure 2 shows a framework for the examination of conceptual understanding developed by teachers.



Note. SC 1 stands for subsidiary concept one, etc.

Figure 2. A macro model of the structure of teachers' mathematical knowledge and problem modelling.

Our model suggests that a conceptually-driven teacher or teaching will place emphasis on developing the knowledge components identified in Figure 2 and the constructions of links among them. For example, in teaching about probability, the teacher will consider not only subsidiary concepts such as fractions but also the many ways to represent the concept of probability. The question is: What is the role of technology in teaching the focus concept, and how could it be effectively used as an instructional tool?

We return to our model in seeking tentative answers to these questions. The effective utilisation of technology as an instructional tool by the teacher could be examined by the actions of the teacher in his/her attempts to foster the construction of understandings that are based on components in Figure 2. A feature of such teaching could be the sequencing of lessons and learning experiences such that students are provided with examples and various images or representations of the focus concept. Additionally, teachers also need to draw out the conceptual building blocks vis-à-vis the subsidiary concepts. Such an approach would enable students to locate the focus concept within their prior knowledge, and develop a better understanding of the focus concept and its relations to other related concepts. The success of the lessons would also be the influenced by the knowledge the teacher has about the students.

The Interviews

In their interviews the teachers certainly did emphasise the links between different representations of function and the role of technology in moving between them. For example the 'obvious' link between the algebra and graphs was mentioned, but their comments on the linking also emphasised the secondary concepts.

> A: ...the roots of the equation how they can be found graphically and algebraically...so that we can move onto parabolas and I will get them to virtually plot the points using a calculator...they won't relate that to any roots of a parabola, that they will think that the graph is a completely separate operation and it's only after they've done it a great number of times that they will see a sort of relationship.

> B: I guess the things that interest me are where they cut the x and y axis. Obviously graphically it is very easy to tell where they are. Besides using a graphics calculator or even using Excel or WL plot, or something like that I can easily check where they cut, so if I was doing this with a class, I think that I would begin without looking at trying to factorise a polynomial and trying to show that where the factors equal 0, there are the roots. I would get the kids to explore where it cuts the y=0 line by getting closer and closer and magnifying, blowing up...I think are a lot clearer to students through using the calculator, graphics calculator or through using a computer program.

Here the teachers both highlight the need to relate algebraic aspects of function, including subsidiary concepts of factor and root, to graphs drawn using a computer or calculator. Other subsidiary concepts of function were similarly prominent in their comments, such as dependent and independent variables, and coefficients, relating the algebra to a graph.

> A: Yes, the idea of bringing in coefficients, the initial worksheet works with the coefficients of x^2 or even earlier back to the straight lines. It's really the idea of the coefficients and the line being to the coefficient of x being the gradient slope...I really think that it is an important thing to get across the formula for the x and y axis, and that actually comes out really nicely on the computer, when you've gone through all of the gradients and they can see the gradients getting closer and closer to 0...I think that's the first time that it becomes clear that as you vary this value, you get a corresponding change in the second variable and the relationship between the two.

> B: I stress the idea of an independent variable and a dependent variable and again I think that something like Excel makes that patently obvious. You know, its sort of reality that here we are choosing particular values and here is what comes out of that choice, so from that point of view I think that's really magic.

The algebraic and graphical representations were not the only ones they referred to, of course. Tables and ordered pairs (or co-ordinates) also featured strongly, with connections made to graphs, etc. Seeing ordered pairs as a separate representation requiring an explicit link to others may be overlooked some times. Comments relating to these included:

> A: ...showing the relationship in ordered pairs and then you can show these and how they relate on a graph...that actually has a formula or a relationship and then we can move on from there to the actual equation

> A: I introduce the difference between functions and relations using the mapping arrow diagrams showing how the value of the range is repeated which means that it's not a function. Similarly the ordered pairs, the member of the range affected and you can show on the graph that when the name's repeated you get points one above the other, and therefore a vertical line test will show that.

> B: I'm impressed by the fact that you can use, develop, a table on a graphics calculator or develop a table on Excel and then just plot those values on a graph. I think it's a very rich step and a step that is lost in using a graphic program.

> B: The other beautiful thing about something like Excel is that you can have, and a graphics calculator, is that you can have a function or a polynomial in different forms such as $x^3 + 6x^2$ + something, and then in a factored form, and by drawing a table of values you can see that they are in fact identical and you can in fact split the polynomial up into sections x+5, x+6 and multiply each of those various columns together so you can actually see the impact of those factors on the overall polynomial as well.

Teacher A's second comment above actually takes a definition of a function and links this focus concept through three representations, set diagram, ordered pairs and graphs. Similarly, Teacher B's second comment shows that when he teaches using a spreadsheet he explicitly links algebraic factors of a polynomial function with a table of values representation. Comments like these demonstrate the flexibility and conceptual nature of the function schemas of these teachers, and the central role of the technology in passing from one representation to another.

It is also worth noting that both of the teachers, A and B, like the teacher Margot that we have previously described, have modelling of realistic situations as a strong focus of their teaching of function.

A: ...say they had three pieces of wood, so they'll have three x's, two x^2 and an x^3 ...The most useful applications I've seen as far as graphs go...are things like the real life situations of graphs, of comparisons of graphs, like putting things, two axes tall or height...like, as an introduction to rates of change...different shaped vessels, how their rate varies if you continually put in a second amount that is the exact same height, varies to the volume, I know those sorts of graphs are good practical examples, it gives some realistic view of these.

B: The whole curriculum is about going towards maths in the real world and within meaningful context, and I think the use of spreadsheet for example provides opportunities for students to explore real relationships between variables. Things like π and circumference and speed, distances, stopping distances, you know, I mean there's heaps of different relationships that can be looked at through a scatter graph for example, right, through, drawing a line through things, through modelling particular relationships... we are often hung-up on giving them a particular equation and asking students to plot points on it, rather than starting the other way and getting students to have points and then trying to find the equation through that.

Teacher A's Lessons

In analysing the actions of Teacher A, who used Excel to teach functions, we were interested to identify concepts and relations that he helped students experiment with prior to and during the lesson. During the first lesson, Teacher A got students to talk about some of the subsidiary concepts before introducing new ones. Table 1 shows the list of concepts that were discussed in the first lesson. All the concepts are clearly necessary and directly related to understandings about functions in general, and linear functions in particular.

In one of the activities, students were given a table with two columns for the x- and y-values. They were then required to look for a *pattern* that showed the relationship between x and y values. Such a relationship was labelled as *relation*, *equation* and *formula*. The teacher explained the how the terms were connected before introducing the notion of functions. The discussion also helped students use two methods to test for functions.

Table 1
Concepts Considered in the Lesson Prior to Using Excel

Focus Concept: Functions	
Subsidiary Concepts	Comments
x- and y- coordinates	Symbols for showing a point within the Cartesian system (x,y).
Domain/Range	How does elements in domain affect values in the Range?, Restricted domain
Ordered pairs	Geometric meaning of the numbers that appear in as an ordered pair
Rule/Relation/Formula	Relations among these
Functions vs relations	Methods of deciding if a relation is a function

The activities of the teacher in lesson 2 were directed at a) reinforcing concepts from lesson 1, b) developing the concepts from lesson 1 further and c) generating alternative representations and procedures related to the focus concept by drawing on the various menus of Excel. In this lesson students were also encouraged to explore the linkages among the focus and subsidiary concepts more dynamically. They were given a prepared Excel sheet, showing three representations of a linear function: algebraic; tabular; and graphical.

Table 2
Concepts Considered in the Lesson Using Excel

Focus Concept: Functions	
Subsidiary Concepts	Comments
General form of a linear function	$y=mx+c$
Coefficient	The meaning of m , gradient
Graphs	Curves and lines
Meanings of letters in $y=mx+c$	Effect of replacing the letters with other letters
Family of functions	Square functions, square root functions
Rule/Relation/Formula/equations	Relations among these
Functions vs relations	Vertical line test; use of zoom menu to locate points of intersection

The reinforcing of ideas from lesson 1 involved attention to symbols and conventions that are used in writing equations for linear functions. This was achieved by asking students to type in the general equation for linear functions in the Excel worksheets ($y = mx + c$).

Teacher A then directed students to replace the available space on the sheet where the letter in the equation was with numbers and observe the effect of their actions on the table and graph that was generated by Excel. During this phase of the teaching, he alerted students to terms such as coefficient, gradient and constant. The focus of his explanation was on the relationship between the letters x and y, ordered pairs and functions. The discussion at this stage also centred on how to use the number in the ordered pairs as a way to test to see if a relation is a function. The above sequence of activities indicate that our teacher was helping the students construct symbolic, tabular and graphical representations of functions.

While the paper and pencil activities in lesson 1 provided an important starting point for students to explore some of the subsidiary concepts, the activities did not allow student visualise the consequences of their actions immediately nor let them test some of their conjectures. Excel permitted the instantaneous observation of such actions. In addition, Teacher A also used Excel to stimulate constructive dialogue among his students. Within the Excel environment mistakes were easily observed by students and corrective action were taken by students independent of the teacher.

Discussion

In this study we have analysed the knowledge base and skills of two experienced teachers in their use of technology. Data showed that the effective utilisation of technology by these teachers is buttressed by a well developed knowledge schema about the core concept and its relations to other concepts. An important character of this schema is how they were able to exploit the facilities offered by technology in promoting conceptual understanding among the students. This fostering of conceptual development and associated procedures about functions was evident in the way the teachers used computers and graphic calculators to construct alternative representation of the focus concept.

Teachers' use of analytic and graphical representations within a spreadsheet such as Excel can help students locate the secondary concepts and establish links to the focus concept. Teacher A, for example, used Excel to show the geometric effect on the line by altering *m* and *c* as variables in the linear equation. While this brought about spectacular changes to what students could see on the computer screen, his lessons did not deviate from highlighting the subconcepts that underpinned his actions, such as variables, ordered pairs and *x* and *y* coordinates. The flexibility with which the experienced teacher is able to introduce technology without losing sight of the associated concepts is a key to a successful lesson.

Underlying the effective use of technology by our teachers is the high level of knowledge they have developed about the technology they chose to use. Our results indicate that teachers need to build sound knowledge about the menus and procedures that are incorporated into the software or the calculator.

How does technology act as a catalyst for inter-representational thinking? One reason may be that technology facilitated the accessing of prior knowledge that may otherwise remain dormant. Lawson and Chinnappan (1994) showed that some students do not access and use knowledge that they have learnt before. This failure to access available knowledge, it was shown, impeded progress students made in solving problems and acquiring new knowledge. Lawson and Chinnappan (1994) suggested that there was a need to develop strategies that would help students search their memory and activate what has been learnt before. We argue that technology may act as a prompt to access what is stored in the long-term memory.

It is also plausible that the use of technology in learning may cause knowledge to be organised around the technology giving qualitatively better schemas and hence foster access to previously-acquired knowledge. This may be because links are better established and reinforced in an environment that is dynamic and allows students to visualise concepts in different ways, across a number of representations.

The results of this study provide tentative support to the argument that experienced teachers use technology with a view to building conceptual understanding among their students. In doing so, a teacher's focus is on disentangling the various components of the concept(s) that are in question. We suggest that future research need to provide more fine-grained analyses of this link between technology use and the construction of links among the concepts, and across the representations.

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