Students' Visualisations of Three-dimensional Shapes

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This paper reports a cross-sectional pilot study of 30 students from Grades 1, 3 and 5, that investigated students' visualisations and representations of three-dimensional shapes. The study describes how students focused on critical and non-critical aspects of three-dimensional shapes and whether any differences exist between students' visual images, verbal descriptions and drawn representations. An interview-based assessment showed that although the quality of some students' visualisations may improve with grade level, non-mathematical qualities feature strongly in most students' visualisations.

In recent years, an increased emphasis on spatial concepts in the primary school mathematics curriculum has aimed at improving students' spatial development and understanding of geometric ideas (Board of Studies NSW, 1999). This change has reflected widespread concern over secondary school students' lack of understanding of key geometric concepts as well as acknowledgement of the intuitive spatial abilities of young children (Owens, 1994). However, there has been insufficient research into the teaching and learning of geometry and a perceived lack of a theoretical framework to underpin this (Pegg & Davey, 1998). More research is needed to improve teachers' knowledge and understanding of students' visualisation abilities particularly because visualisation is essential for both spatial concept development and spatial problem solving.

Research on Students' Visualisation Abilities

Several researchers have focused on the ways students use mental imagery, or visualisation, in dealing with three-dimensional objects (Battista & Clements, 1991; 1996; Fennema & Tartre, 1985; Owens, 1994, 1999; Pegg, 1997; Presmeg, 1986). Owens (1994) showed that visual imagery was important in young students' noticing features of shapes and in deciding how shapes could be used. This helped students to form geometric concepts.

Fennema and Tartre (1985) found that students who differed in spatial visualisation skills did not differ in their ability to find correct problem solutions, but they concluded that an emphasis on spatial visualisation skills will improve mathematics learning (Fennema & Tartre, 1985, p. 203). Presmeg (1986) discovered that while visual imagery did assist many students in solving problems, visualisers could experience some disadvantages, such as one-case concreteness of image or diagram, inflexible thinking, vagueness with concepts or difficulty in communicating concepts.

Battista and Clements (1991) studied students' responses when using LOGO to draw geometric figures and noted that spatial visualisation may be a more important factor in geometric problem solving for students who are at the visual level (in terms of van Hiele's hierarchy) than at the higher levels of geometric thinking. However, they pointed out that "visual imagery, when properly developed, can make a substantial contribution at all levels of geometric thinking" (Battista & Clements, 1991, p. 20). In a later study, Battista and Clements (1996) found that some students had considerable difficulty in dealing with 3D cube arrays due to their inability to visualise and enumerate the cubes in such arrays. They indicated the importance of further research into the nature of visualisation, particularly visualisation of three-dimensional shapes.

Theories on the Development of Spatial Concepts

Theories about how students develop spatial concepts highlight the importance of the visualisation process in dealing with both two-dimensional and three-dimensional shapes. Visualisation of objects is an important mathematical process which is linked to growth in mathematical development. Davis, Tall, and Thomas (1997) outline some of the conceptual development theories that deal with the transition from *process to object* in mathematical thinking (Davis, 1984; Dubinsky, 1991; Gray & Tall, 1994; Greeno, 1983; Piaget & Inhelder, 1971). Davis, Tall, and Thomas (1997) assert that although the various *stage* theories about spatial development do not necessarily correspond, a commonality can be observed.

> Each student passes through a development of growing sophistication from some kind of process/procedure, usually performed step-by-step, and ending with an object that can be manipulated as an entity in its own right (Davis, Tall, & Thomas, 1997, p. 134).

Recent work by Gray, Pitta, and Tall (1997) has shown that students with poor visual skills may focus on non-mathematical aspects of shapes and this may inhibit effective learning of geometric ideas. Their related research on elementary number development suggested that children interpret information from the world as objects with various properties. Thus the particular *object*, of which a student chooses to create an image, affects the quality of the image formed. The present study investigates Gray, Pitta, and Tall's assertion by examining the quality of students' visualisations, and by describing which aspects students focus on when they form images of three-dimensional shapes. It is critical to ascertain whether young students focus on mathematical properties such as the six congruent faces of a cube.

Battista (1994) supports the notion of investigating mathematical properties of objects through mental models. He refers to Kosslyn's assertion that "one of the most striking things about objects in images is how they mimic properties of real objects" (Kosslyn, 1983, quoted in Battista, 1994, p. 91).

The van Hiele (1986) theory comprising five differentiated levels of geometric thinking distinguished how students operated on shapes and other geometric configurations as visual wholes. However, the students did not explicitly focus on geometric properties. Pegg and Davey (1989) concluded that a great number of students between Year 3 and Year 7 describe a geometrical figure in terms of one property, namely, the number/equality of sides. They also found that "the ability to draw correct diagrams stems from images that students possess and often these images do not reflect student understandings in terms of the properties of a given figure" (Pegg & Davey, 1989, p. 25).

The early development of geometric concepts is based on the images students create, thus it is critical to examine the way these images are formed and used. It appears that in order to better understand and assist young students with the development of spatial concepts, there is a need for further investigation of students' understanding of the properties of three-dimensional shapes. If it can be ascertained what students choose to focus on when analysing shapes, then teaching programs can be developed to improve students' attention to critical aspects.

This study aims to address three key research questions:

1. How well do students visualise three-dimensional objects?
2. In their visualisations, do students focus on critical or non-critical aspects of three-dimensional objects? Are these aspects mathematical properties?

3. What are the differences, if any, between students' visual images, verbal descriptions and drawn representations of three-dimensional objects?

Method

A cross-sectional descriptive study was considered appropriate so that students' visual imagery could be examined as closely as possible.

Table 1
Assessment Tasks

Task 1: Visualise a three-dimensional shape
I want you to think about a cereal box, for example, a cornflakes or rice bubbles box. Tell me all you can about the shape of this box.
Task 2: Identify similar shapes
Can you think of any other things or shapes in the real world that are the same shape as this cereal box shape? Why are they the same?
Task 3: Name the mathematical shape visualised
Do you know the mathematical name of this cereal box shape?
Task 4: Draw the visualised shape
Can you draw this cereal box shape for me? Can you explain your drawing to me?
Task 5: Describe a (held) shape
I've got a real cereal box here. You can pick it up and turn it around if you want to. Now can you describe the shape of this box to me? (If the description was quite different from the original visualisation, the investigator said "You said 4 sides before, and now you have told me there are 6 sides. Why did you say 4 sides before? How was the picture in your mind different from the real box?")
Task 6: Identify shapes needed to make up into three-dimensional shape
Here are some cardboard shapes. If you wanted to stick some of these shapes together to make one cereal box, which shapes would you need? Hand them to me.
Task 7: Identify net of a shape
This is the shape of a cornflakes' box flattened out. Now if you cut out these shapes (paper with nets A- E was shown) and folded them up along the dotted lines, which ones could you make into a small cornflakes box shape?
Task 8 : Describe a blank (held)shape
Now look at this box (child was shown a muesli bar box which had been folded inside out so that all faces appear blank to avoid distraction. The student was allowed to handle the box for a few seconds. Then it was taken from view). Now describe the shape of the box.

An interview-based assessment of students' understanding and visualisations of three-dimensional shapes included eight tasks (see Table 1) adapted from instruments used in prior studies (Battista & Clements, 1996; Shaughnessy, 1999, in correspondence). The tasks also reflected sample activities from the NSW Mathematics K-6 syllabus (NSW Department of Education, 1989).

A sample of 30 students, ten from Year 1, nine from Year 3 students and eleven from Year 5 was drawn from a NSW Department of Education and Training school in a lower socio-economic area of Sydney. Interviews were administered on a one-to-one basis by the chief investigator. During the interview the students were given the opportunity to represent their visualisations in a variety of ways before and after they had observed and manipulated actual shapes. Responses were recorded on an audiotape and students' drawings and explanations were retained for later analysis. Solution methods were coded for correct, incorrect, or non-response before being analysed for key mathematical aspects. A coding scheme was devised and later revised after analysis had been complete. Coding of responses was supervised and re-coded by an independent coder. Classification of responses reflected 80% agreement between coders for the purpose of initial pilot analysis.

Discussion of Results

Interview transcripts were analysed and responses classified according to (i) student performance, (ii) the mathematical or non-mathematical aspects of the responses and (iii) differences between drawn, visualised and verbal descriptions.

Table 2

Percentage of Students' Responses by Category and Year Level for Tasks 1-8

	Year 1 n = 12	Year 3 n = 11	Year 5 n = 11
TASK 1: visualise three-dimensional shape			
described shape using non-math props only	17%	0%	0%
described shape using non-math and math props	83%	80%	73%
described shape using math props only	0%	20%	27%
unable to name any maths props correctly	92%	60%	9%
name one prop correctly	8%	30%	27%
name two props correctly	0%	10%	55%
name three props correctly	0%	0%	9%
made incorrect estimate of either faces, corners or edges	50%	70%	36%
TASK 2: identify other things with the same shape			
unable to name anything with similar shape	33%	30%	18%
named one other thing with the same shape	58%	40%	45%
named more than one thing with the same shape	8%	30%	36%
TASK 3: name the mathematical shape visualised			
gave correct math name for shape	0%	20%	36%
TASK 4: draw visualised shape			
drew shape as 2D drawing	80%	30%	9%
drew shape as poor 3D drawing	20%	45%	36%
drew shape as a good 3D drawing	0%	25%	27%
TASK 5: describe a held shape			
described shape using non-math props only	17%	0%	18%
described shape using non-math and math props	83%	90%	82%
described shape using math props only	0%	10%	0%
unable to name any maths props correctly	75%	10%	9%
name one prop correctly	25%	70%	55%
name two props correctly	0%	20%	27%
name three props correctly	0%	0%	9%
made incorrect estimate of either faces, corners or edges	8%	50%	45%
TASK 6: identify shapes needed to make up into 3-D shape			
chose 6 correct shapes	0%	30%	36%
chose 6 shapes but incorrect ones	8%	20%	36%
chose 4 shapes only	33%	30%	18%
chose other incorrect combination of shapes	58%	20%	9%
TASK 7: identify net of shape			
identify correct nets	0%	10%	0%

TASK 8: describe "blank" held shape			
described shape using non-math props only	8%	0%	0%
described shape using non-math and math props	92%	100%	91%
described shape using math props only	0%	0%	9%
unable to name any math props correctly	83%	30%	9%
name one prop correctly	17%	50%	36%
name two props correctly	0%	20%	45%
name three props correctly	0%	0%	9%
made incorrect estimate of either faces, corners or edges	75%	70%	67%

The results of the analysis are shown in Table 2 above. In the discussion following, the research questions will be addressed in turn. Excerpts are used to elucidate some distinctions between students' responses.

Overall, students found difficulty in visualising three-dimensional objects with an accurate awareness of their mathematical properties. Table 2 shows that for Task 1 most children could visualise a three-dimensional object, but could not describe the three-dimensional aspects of it. Results from Task 2 showed that a number of students were unable to match their visualisation of a cereal box shape to any similar shape in the real world (33% in Year 1, 30% in Year 3 and 18% in Year 5). Some of the responses given by students were related to a two-dimensional rather than a three-dimensional image eg "a door", "a piece of paper".

Results for Task 3 indicate that most students were unable to give the visualised box shape its correct mathematical name and even those students who did name the shape correctly in Task 3, sometimes misnamed it in later tasks. It seems that, considerable confusion existed at all grades about the correct meaning of the terms *side, face, corner, edge, rectangle,* and *square* as well as *rectangular prism.*

Task 6 revealed that most students had difficulty selecting the cardboard shapes that would make up into a cereal box shape. Table 2 shows that only 36% of Year 5 and 30% of Year 3 students completed the task accurately. The large number of students (a total of 91% for Year 1, 50% for Year 3 and 27% for Year 5), who choose either four shapes only or any other incorrect combination of shapes (i.e., not six shapes) reinforces results of Task 1 where students were unable to describe the number of faces of the cereal box shape. Although Task 6 was completed after the students had handled a cereal box in Task 5, it required a good deal of visualisation ability to imagine the cardboard shapes, which were laid flat on the table, built up and stuck together as a single three-dimensional shape. It appears that the visualisation ability necessary to complete this task had not yet developed for the majority of students.

Task 7, in which students were asked to examine five possible nets of a cereal box shape and identify the correct nets, was poorly answered by all students. This task required more developed visualisation ability than for Task 6 as the students had to imagine all the faces folded up without the opportunity to handle the individual faces.

2. In their visualisations, do students focus on critical or non-critical aspects of three-dimensional objects? Are these aspects mathematical properties?

The results showed that non-mathematical aspects featured strongly in students' responses across grade levels. Overall, a majority of students used a combination of non-mathematical and mathematical properties. Table 2 shows that for Task 1, 17% of Year 1 students described the shape using non-mathematical properties only. These students talked about the colour, the illustrations and the texture of the cereal box shape, but did not use mathematical terms such as *faces, vertices* or *edges*. For example: Mark (Year 1): "It has pictures on it, it's fat and sort of hard and it's got red and white writing on it".

A further 83% of Year 1 students, 80% of Year 3 and 73% of Year 5 students described the shape using a combination of non-mathematical and mathematical properties. However, 20% of Year 3 and 27% of Year 5 were able to describe the shape using mathematical terms only, even though at times the mathematical descriptions were not always correct. For example: Toa (Year 5): "It's a rectangle and it's got 6 faces and 4 sides."

Similar results appear for Task 5, when the students were asked to describe the shape of a cereal box they were given to hold. Again most students used a combination of non-mathematical and mathematical properties in their descriptions (83% in Year 1, 90% in Year 3, and 82% in Year 5). The attention to non-mathematical properties in Tasks 1 and 5 continued despite the researcher's prompt, "Is there anything else you can tell me about the shape of the box" (with emphasis on the word *shape)*. Similarly in Task 8, despite the removal of stimuli, students continued to mention the colour, design and texture of the shape as well as some mathematical properties. For example: Amber (Year 1): "It's brown and it feels rough and it has sticky tape on it and it has four corners and it's a square".

Responses to Tasks 1, 5, and 8 which described the shape using mathematical terms only were low (0% Year 1, 20% Year 3 and 27% Year 5 respectively). In their visualisation for Task 1, most Year 1 students (92%) were unable to name correctly any mathematical properties of the shape. With the box held in their hands, 75% of these Year 1 students were still unable to name correctly any mathematical properties of the shape. Does this imply that their visualisation of the shape was incorrect, or does it reflect their unfamiliarity with correct mathematical terms and properties? Similarly, in Year 3, 60% were unable to name any mathematical properties in describing their initial visualisation. However, with the shape held in their hands for Task 5, only 10% were still unable to name any properties. It seems likely that by Year 3 these students had had greater exposure to teaching about the mathematical properties of rectangular prisms and this teaching was reflected in their answers when the shape was held. The fact that these properties are not mentioned in their descriptions of their visualisations seems to indicate that their visualisations of the box shape may indeed be flawed.

By Year 5, only 9% of students were unable to name any mathematical properties both in their description of their visualisation of the box (Task 1) and their description of the held shape (Task 5). However, it cannot be assumed that visualisation ability has improved for all students. This may be the case for some, although 36% of these Year 5 students were still making an incorrect estimate of the number of either faces, or vertices, or edges in their description of their visualisation.

3. What are the differences, if any, between students' visual images, verbal descriptions and drawn representations of three-dimensional objects?

In general, it seemed that there was no relationship between students' visualisations, their verbal descriptions and their ability to draw a three-dimensional shape. Indeed there may be considerable differences between students' abilities on these three aspects. Results for Task 4 showed that students' ability to produce a drawing that was a good representation of a three-dimensional shape improved markedly with grade level. Most of the Year 1 drawings consisted of a simple two-dimensional shape, often including illustrations or an attempt at words. By Year 3, 65% of students were producing a drawing that indicated the shape had a three-dimensional quality, although most of them were still including illustrations or words. By Year 5, only 9% of students, compared with 80% of Year 1 students were still drawing the cereal box they had visualised as a simple two-dimensional rectangle or square. Most drawings of Year 5 students indicated that the shape had a three-dimensional quality. Whether it was rated a poor or good three-dimensional drawing reflected only the students' individual drawing skills. However the ability to draw a good three-dimensional figure of a cereal box shape did not necessarily mean that the student had an accurate visualisation of that same cereal box shape. For example, Amanda (Year 5) was able to draw a fairly good representation of a cereal box shape indicating its three-dimensional nature. Yet in her description of her visualisation, Amanda said: "It's a rectangle at the front and back and on the side it has strip things. The top opens and it has a plastic packet with cereal inside and there's writing on it." Amanda's language, while containing mathematical terms, is far from accurate. Jeremy (Year 5), on the other hand, drew a simple two-dimensional representation, but his visualisation was more accurate: "It's like a 3D shape. Sometimes it has a bowl of cornflakes on it and writing and other stuff. It's got 6 faces and 8 corners." It would appear therefore that the accuracy of drawing a three-dimensional shape which a student has just visualised does not necessarily reflect the student's visualisation ability.

Implications

This study has highlighted the difficulties that many young students may experience in visualising three-dimensional shapes. Gray, Pitta, and Tall (1997) indicated the need to examine the quality of the object which dominates the child's visualisation. This study supports their findings that non-mathematical or non-critical qualities appear to dominate, or at least feature strongly, in students' visualisations of objects at the expense of mathematical properties. It also appears that the quality of some students' visualisations may improve with grade level, but that students may remain focused on non-mathematical or non-critical aspects of shapes. The study found that students' drawings of the visualised shapes also may, or may not, reflect the description of the visualisation. In addition, students' descriptions of their visualisations of three-dimensional shapes do not necessarily match with their description of the same shape after handling it. This relates to Owen's (1999) notion that verbal expressions are associated with visual imagery and help to define it. Owens found that as students group and regroup shapes, they refine their images of these shapes.

It may be advantageous to analyse students' visualisations of shapes according to qualitative differences in their attention to mathematical and non-mathematical properties. Levels could be assigned which may reflect the van Hiele (1986) levels of cognitive development, but this may produce a stage-like model that detracts from the rich and complex development of the individual. Southwell (1997) also discusses the use of the van Hiele levels of geometric development either to establish a curriculum or to ascertain the level of understanding at which students are operating. Southwell suggests that both aspects may be possible, but when using van Hiele levels as an assessment instrument, difficulties may arise when a student operates at different levels for different aspects of spatial thinking. Pegg and Davey (1998) highlight the importance of "identifying individual paths of development: to seek out variability" (p. 133) rather than using van Hiele's unidimensional descriptive model of learning. The present study indicates that students' visualisation ability is fundamentally formed at an early level where imagery plays a critical role. Investigating students' use of imagery might then be considered an essential element of any overall assessment of spatial development. The study also confirms Battista and Clements' (1996) suggestion of the importance of further research into the nature of visualisation, particularly the visualisation of three-dimensional shapes.

Owens (1999) developed a framework which showed how visualisation strategies are central to the development of spatial thinking. Further research needs to use a framework as a basis for designing a teaching program aimed at improving students' visualisation ability of three-dimensional shapes. This research could assist in establishing how practical activities could improve visualisation skills. Such activities might involve handling and describing three-dimensional shapes, comparing and contrasting the properties of such shapes, folding and unfolding nets to make up three-dimensional shapes and reproducing shapes from memory. Further study is warranted to examine the relationship between improving students' use of correct names and properties of three-dimensional shapes, and their visualisation ability. Classroom studies are needed to ascertain whether improved teaching and assessment might lead to a much closer match between students' visual images, verbal descriptions and drawn representations of three-dimensional images.

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