Understanding of Functions Among Maldivian Teacher Education Students

Ibrahim Hassan
Maldives Institute of Teacher Education
<chubby.ih@eudoramail.com>

Khadeeja Mohamed
Maldives Institute of Teacher Education
<kadda@mesco.com.mv>

Michael Mitchelmore
Macquarie University
<mike.mitchelmore@mq.edu.au>

Maldivian teacher education students' understanding of functions and rates of change was investigated. The written responses of 22 students to 10 examination questions were first analysed, and then 5 students were interviewed in depth as they solved 4 of the most difficult questions. It was found that students had a good grasp of basic concepts, but tended to rely on remembered formulae or procedures. They also had difficulty visualising tangents and linking them to rates of change and derivatives. Implications for teaching are explored.

The National Diploma of Secondary Teaching (NDST), a tertiary program which prepares teachers for Maldivian lower secondary schools (Grades 8-10), started at the Institute of Teacher Education (ITE) in February 1997. The two-year program combines academic courses in mathematics, various sciences, and education with field-based teacher education. The mathematics and mathematics education elements have been developed and taught by the first two authors in consultation with the third author.

Students enter the NDST with London A-level passes in at least two subjects. However, experience suggests that most students have achieved these passes largely through rote learning. Regarding this as an inadequate basis for teaching, the designers of the NDST aimed to develop a program which would strengthen students' understanding of elementary concepts as well as teaching more advanced content. For example, the first-semester mathematics course MATH101 includes a 5-week module on Functions and Rates of Change which seeks to put calculus techniques learnt in school on a firm conceptual footing. It was important for us to ask the question: How effective is this module?

Functions and Rates of Change

The apparent tendency of calculus courses to promote rote, manipulative learning has been a subject of concern for more than a decade. We only have space to cite a few related research studies: Poor understanding of the underlying concept of function has been shown among mathematics students (Eisenberg, 1991), teacher education students (Even, 1993) and teachers (Cooney & Wilson, 1993). The concept of rate of change causes particular difficulties (White & Mitchelmore, 1996). Most recently, Santos-Trigo (1998) analysed errors among first-year calculus students and found that several were due to inadequate mastery of elementary concepts such as use of functional notation, evaluation of limits, and a restricted tangent concept.

There have been numerous curriculum responses to what some have called a crisis in calculus teaching (White, 1990). Some authors (Barnes, 1992) have used everyday situations as the basis for developing students' intuitions about functions, following an approach similar to that used by the Dutch *Realistic Mathematics Education* movement (Streefland, 1991). Others have focussed on the relation between functions and graphs, often using computers or graphics calculators to assist and develop visualisation (Keiran, 1993). Still others have emphasised the importance and value of being able to translate fluently between different representations of a function, another area where technology can be particularly helpful (Kaput, 1992).

The MATH101 module on Functions and Rates of Change was developed with the above research and development in mind. The teaching approach adopted is well exemplified in Chapters 1 and 2 of the chosen textbook (Hughes-Hallett et al., 1994): Students sketch graphs of realistic functions, learn to recognise the graphical forms of a basic set of functions (polynomials, exponential and logarithmic functions, etc.), measure rates of change in various ways, and sketch graphs of rates of change. The textbook strongly emphasises what it calls "The Rule of Three: Every topic should be presented geometrically, numerically, and algebraically" (p. vii). We also use Barnes (1992) to provide additional realistic examples of functions and rates of change. Unfortunately, no technological aids are available at the present time.

Despite the efforts which have gone into designing MATH101 (which also includes modules on Vectors & Matrices and Number Theory), examination results have been rather disappointing. After an analysis of the archived examination scripts showed poor performance on the Functions and Rates of Change items (Hassan, Mohamed, & Mitchelmore, 1999), a small number of students was interviewed as they solved related items. The present paper presents an error analysis of the examination scripts and the interview responses. The primary aim of the error analysis is, as suggested by Fong (1995), to identify students' underlying conceptual difficulties and thus to obtain guidelines for improving the teaching of the module.

Script analysis

Method

A total of 27 students have taken the MATH101 examination since the inception of the NDST: 5 in 1997, 10 in 1998, and 12 in 1999. Because 1997 was the first year of the program and involved a small number of students, the 1997 scripts were excluded.

There was a total of ten items on Functions and Rate of Change on the 1998 and 1999 papers. One was rather misleading, and two were completed correctly by almost all students. The other seven are shown in abbreviated form in Figure 1. Items 3 and 4 appeared in identical form in 1998 and 1999. Items 6 and 7 appeared in 1998 and similar items in the 1999 examination. The other three items appeared in one year only.

Results

In the following, a notation such as "(8/12)" indicates that 8 out of 12 students exhibited a particular error at least once.

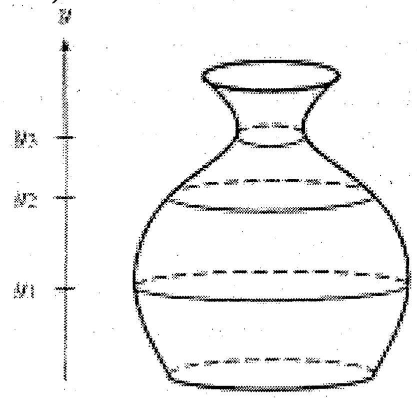
Item 1. Most students (10/12) failed to answer this item correctly. Most of these students had clearly misinterpreted the graph as a graph of displacement against time: They chose a graph which was increasing but concave downward instead of upward, claiming that athletes tend to slow down as they become tired during a race.

1. Which graph models most realistically the relationship between the time it takes to run a race and the length of the race? Explain your choice. If you think none of the graphs is realistic, draw your own version and explain it fully. [Six graphs were given, each showing time as a monotonically increasing or decreasing function of length.]

2. Find a possible formula for the graph below. [The graph of a monotonically decreasing, apparently exponential, function through (-1, 8) and (1, 2) was given.]

3. The depth of water in a tank oscillates sinusoidally once every 6 hours around an average depth of 7 metres. If the smallest depth is 5.5 metres and the greatest depth is 8.5 metres, find a formula for the depth in terms of time, measured in hours. (Sinusoidally is another name for sine curve).

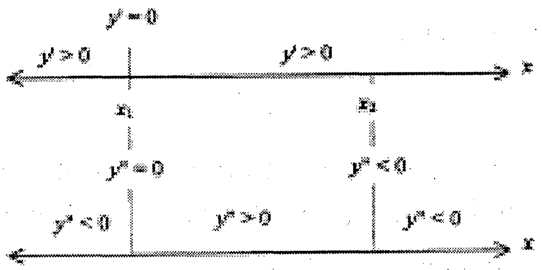
4. Consider the vase drawn below. Assume the vase is filled with water at a constant rate (i.e., constant volume per unit time).



(a) Graph $y = f(t)$, the depth of the water, against time, t. Show on your graph the points at which the concavity changes.
(b) Where does $y = f(t)$ grow fastest? Slowest? Estimate the ratio between these two growth rates.

5. Sketch the graph of $y = f'(x)$ for the function given below. [A continuous graph was given with x-intercepts at 2 and 4, a maximum at $x = 0$, a minimum at $x = 3$, and points of inflection at $x = -2$ and $x = 4$.]

6. Sketch a possible graph of $y = f(x)$, using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. Assume that the function is defined and continuous for all real x.



7. Identify the local and global maxima and minima of $f(x) = x^{-2/3} + x^{1/3}$ ($1.2 \le x \le 3.5$.

Figure 1. Examination items on Functions and Rate of Change module.

Item 2. All students recognised the given function as possibly exponential but most (6/10) could not, for various reasons, fit an appropriate equation to it. Most of these (5/10) did obtain an equation, but they failed to check that the function so defined passed through both the given points.

Item 3. Most students (16/22) failed to answer this item correctly. Five students omitted the item altogether, and the remainder made a variety of errors (mostly involving the amplitude and the mean height). There was little evidence of students attempting to check their equations—for example, by sketching a graph.

Item 4. All students (22/22) failed to predict the shape of the graph precisely. The main error consisted of assigning the second point of inflection to the height y_2 rather than y_3 — although most students correctly stated that the maximum rate of change is at y_3. All students (22/22) also failed to estimate the ratio of the minimum to the maximum rate of change on Item 4. Nine students omitted this part completely, whereas the remainder appeared to use the diameters of the vase at the heights y_1 and y_3 rather than the cross-sectional areas.

Item 5. Half the students (6/12) made errors responding to this item. All these students drew a graph of $f'(x)$ which had a maximum at $x = 5$ instead of $x = 4$. (On the given graph $f''(x)$ had a minimum at $x = 5$, giving a point of maximum curvature.)

Item 6. This item was difficult in both versions, with almost all students (21/22) making some error. Students had slightly more difficulty interpreting the information about the first derivative (18/22) than the second derivative (15/22).

Item 7. On both versions of this item, most students (19/22) made errors. On the item shown in Figure 1, most students (8/10) had difficulty with the rational powers: Some made errors in the derivative, some were unable to solve the equation to find whether the derivative is zero, and some were unable to calculate the value of the function at the end-points. Several students were unable to reconcile the local minimum they obtained with the values of the function they calculated at the end-points of the given interval. On the other version of Item 7, in which $f(x)$ was a cubic polynomial defined over all real numbers, most students (10/12) calculated the local maximum and minimum correctly and then assumed these values also gave the global maximum and minimum.

Summary

The errors students made on the examination scripts suggested that they were not sufficiently skilled in the following aspects of graphical reasoning:

1. From a description of a physical situation:
   (a) identify two variables, one of which is varying as a function of the other;
   (b) represent the general form of their relationship in a graph.
   (The term "general form" includes whether the function is increasing or decreasing over the domain, whether it reaches a maximum and then decreases, whether the rate of change is constant, and so on.) This skill was essential for Items 1, 3 and 4.

2. Sketching a graph by finding *precisely* where its gradient changes from increasing (becoming steeper if positive or less steep if negative) to decreasing:
   (a) from a description of a physical situation; or
   (b) from information about the first and second derivative.
   This skill was essential for Items 4, 5 and 6.

3. Using a graph as a representation of a function over its entire range of definition, for example:
   (a) to check whether a given equation models the function adequately, or
   (b) to find the range of variation of the function.

This skill was important in Items 2, 3 and 7.

The interviews were accordingly designed to learn more about students' understanding of these aspects of graphical reasoning.

Interviews

Method

It was decided to interview students as they solved four of the most difficult examination items, namely Items 1, 3, 4 and 6. Each item was presented on a separate sheet of A4 paper, on which students also wrote their working. Item 1 was re-worded to require the student to draw a graph rather than to select a graph from several. An additional easy item (finding the equation of a line through two given points) was included as a warm-up to help students learn the technique of "thinking aloud".

Five first-year NDST students, all of whom had taken the MATH101 examination approximately 12 weeks earlier, were interviewed. These 5 students were those judged by their lecturer to be most likely to be able to think aloud. All the students were female.

The students were each interviewed individually for about an hour by the first and third author. Whenever students forgot to talk about what they were doing, the interviewers asked students to describe what they were thinking or trying to do. The interviewers also frequently asked students to explain what their drawings or writings meant and how they had worked them out. When students could not quite complete a task, the interviewers occasionally asked leading questions to help the students reach a more complete understanding of the item.

For technical reasons, the interviews could not be recorded. The data collected for analysis consisted of the students' written work together with the interviewers' notes on (a) their observations of the students' behaviour during the solution process (e.g., long pauses, apparent visualisation movements) and (b) students' answers to the probing questions.

Results

Item 1. All 5 students correctly identified the two variables, explained the concepts of dependent and independent variables, and identified "the time taken to run a race" as depending on "the length of the race". However, none was able to successfully represent this relation graphically. Most students started by drawing a graph of length against time; only 3 attempted to draw a graph of time against length. Two students plotted two pairs of values (10 s for 100 m and 20 s for 200 m), but could not decide whether 200 m would take proportionately longer than 100 m or what this would imply for the shape of the graph. All students interpreted the relation as referring to a single race (i.e., a distance-time graph) and spoke about "the speed at which he runs". There could have been some linguistic confusion; one student said "length means distance" and another said "ah, 'time it takes' means the time as someone runs". But it seemed to the interviewers that the basic problem was that students expected graphs to show a relation between variables which actively change, rather than simply having a range of possible values. As one student asked, "Are we talking about *change* or *relationship*?"

Item 3. Although all the students had great difficulty demonstrating the physical motion which this question describes, they all correctly drew a horizontal time axis, a vertical depth axis, and a sine curve with period 6 hr oscillating between 5.5 m and 8.5 m. (The time which some of them required to do this explains why no student drew the graph under examination conditions.) But only 3 of the 5 students were able to correctly calculate or interpret the average depth of 7 m. The same 3 students were also the only ones who were able to calculate the amplitude as the maximum deviation from the average and to construct the correct formula $d = 7 + 1.5\ \sin(\frac{\pi}{3}t)$. However, all 3 seemed to be working from the memorised formula $d = A \sin Bt$, where A = amplitude and $B = 2\pi \div$ period, together with a vertical translation by 7 m. No student attempted to visualise a transformation from $d = \sin t$ into $d = 1.5\ \sin(\frac{\pi}{3}t)$ or to find the coefficients A and B by substituting appropriate values of t.

Item 4 . All 5 students were able to sketch the general form of the graph, to explain the variation in slope, and (when pressed) to identify the precise heights at which the water level was rising fastest and slowest. However, as in the written examination, several students initially identified a range of depths (usually between y_2 and y_3) where the rate of change was greatest, instead of the single point y_3. Students' explanations suggested that most were thinking in terms of average rates of change rather than instantaneous rates of change. For example, one student marked about 10 intervals on the t-axis and said "In each second, water has to spread out different amounts"; another divided the vase into horizontal layers and said "It takes longer to fill up the wider layers". Also, all students had difficulty drawing tangents at the points of inflection; they would repeatedly draw the tangent at a nearby point, even though they realised the gradient was not as steep there (see Figure 2). Students admitted to believing that a tangent must "touch" the curve, and therefore should not cross it. Students knew the term "point of inflection" and knew that the concavity changes at such a point, but they appeared not to have thought of it in terms of the changing slope of the tangent.

Item 6. All students showed a very weak concept of derivative. Only one student immediately interpreted the first derivative y' as the gradient and used it correctly to sketch the graph. Three students stated that $y' = 0$ meant the graph had a turning point at x_2 and $y' > 0$ meant that the gradient was positive; two managed to resolve this contradiction and sketched a horizontal point of inflection at x_1, but the third could not. The fifth student could not even start to answer the question. Students' concepts of the second derivative were even weaker. Three students said they only knew that you calculate y'' to find whether a turning point is a minimum or a maximum; two of these also remembered that $y'' > 0$ means the gradient is increasing and were then able to deduce that there is a point of inflection at x_1. No student was able to say what y'' meant, or why it was called the *second* derivative.

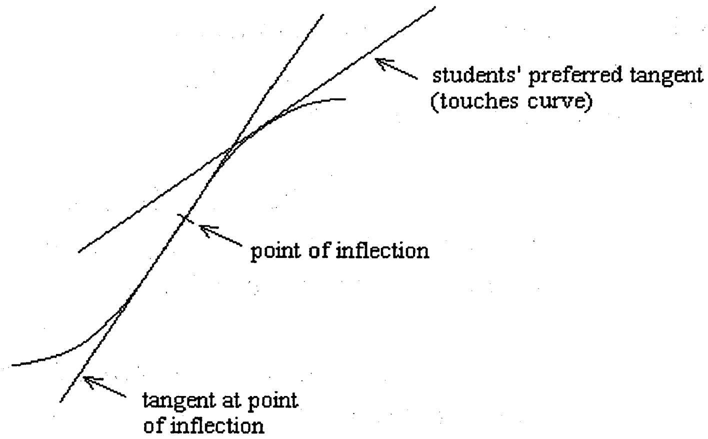


Figure 2. Drawing the tangent at a point of inflection.

Summary

The individual interviews confirm the findings of the script analysis and are generally in line with previous research on students' concepts of function and rate of change. Students knew the names of some important concepts (e.g., function, second derivative, point of inflection) and could identify examples of them, but they were often unable to relate them in the manner required in graphical reasoning. Like Australian students (White & Mitchelmore, 1996), they tended to rely on formulae and rules, and they were severely restricted by their own experience (Goldenberg, 1987). They found it difficult to visualise a tangent moving along a curve, seemed not to distinguish average from instantaneous rate of change, and did not appear to have formed strong links between the concepts of tangent, derivative, gradient, and rate of change (Santos-Trigo, 1998).

Implications

Functions feature widely in the Maldivian school mathematics syllabus, but the emphasis is on algebraic manipulation. In that respect, the approach adopted in MATH101 is quite new and some of the errors we observed may have been due to this novelty. In general, however, our findings suggest that many students have not grasped several concepts which are fundamental to the tertiary study of functions.

We deduce that it is appropriate to include the module on Functions and Rates of Change in MATH101 unit and that no essential changes need to be made. However, there are clearly some aspects which need extra emphasis if it is to be as effective as possible:

- When sketching everyday functions (as in Barnes, 1992), challenge the idea that a graph is not necessarily a static or dynamic picture of actual events.
- Emphasise even more the relation between rate of change, derivative, and gradient. Distinguish more clearly between average and instantaneous rates of change. Do not assume that students have mastered these concepts before commencing tertiary studies.

- Focus more on how the rate of change of a function itself changes. Challenge the idea that a tangent "touches" a curve without crossing it. Obtain technology to "draw" tangents and show how they vary along a curve. Emphasise precision in locating the point on a curve where the concavity changes.
- Emphasise that the second derivative is the rate of change of the first derivative. Sketch graphs of both first and second derivatives of given functions (and vice versa). Derive logical explanations for rote-learnt rules for using the second derivative to classify turning points.
- Hold even more strongly to the "rule of three" by continually reminding students of the arithmetical, algebraic, and graphical representations of mathematical ideas and the links between them.

Our findings have implications for the teaching of mathematics in Maldivian schools. We hope they will also be of interest to Australasian mathematics educators.

References

Barnes, M. (1992). *Investigating change*. Melbourne, VIC: Curriculum Corporation.

Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about functions: Historical and research perspectives. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 131-158). Hillsdale, NJ: Erlbaum.

Eisenberg, T. (1991). Functions and associated learning difficulties. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 140-152). Dordrecht, The Netherlands: Kluwer.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24,* 94-116.

Fong, H. (1995). Schematic model for categorising children's errors in solving a ratio and proportion problem. *Hiroshima Journal of Mathematics Education, 3,* 15-29.

Goldenberg, E. P. (1987). Believing is seeing: How preconceptions influence the perception of graphs. In J. C. Bergeron, N. Herscovics & C. Keiran (Eds.), *Proceedings of the 11th International Conference for the Psychology of Mathematics Education* (pp. 197-203). Montreal, Canada: Program Committee.

Hassan, I., Mohamed, K., & Mitchelmore, M. C. (1999). *Mathematical thinking among Maldivian teacher education students.* Unpublished report, Macquarie University.

Hughes-Hallett, D., Gleason, A. M., Flath, D. E., Gordon, S. P., Lomen, D. O., Lovelock, D., McCallum, W. G, Osgood, B. G., Pasquale, A., Tecosky-Feldman, J., Thrash, J. B., Tucker, T. W., & Quinney, D. (1994). *Calculus*. New York, NY: Wiley.

Kaput, J. J. (1992). Technology And Mathematics Education. In D. A. Grouws (Ed.), *Handbook Of Research On Mathematics Teaching And Learning* (Pp. 515-556). New York: Macmillan.

Keiran, C. (1993). Functions, Graphing, And Technology: Integrating Research On Learning And Instruction. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating Research On The Graphical Representation Of Functions* (Pp.189-238). Hillsdale, Nj: Lawrence Erlbaum.

Santos-Trigo, M. (1998). Patterns of mathematical misunderstanding exhibited by calculus students in a problem solving course. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 89-96). Stellenbosch, South Africa: Program Committee.

Streefland, L. (Ed.). (1991). *Realistic Mathematics Education In Primary School.* Utrecht, The Netherlands: Cd-ß Press.

White, P. (1990). Is Calculus In Trouble? *Australian Senior Mathematics Journal, 4,* 105-110.

White, P., & Mitchelmore, M. C. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education, 27,* 79-95.