A Framework for Teaching Early Measurement

Lynne Outhred
Macquarie University
<lynne.outhred@mq.edu.au>

Diane McPhail
NSW Department of Education and Training
<diane.mcphail@det.nsw.edu.au>

This paper describes a framework for teaching length, area and volume in the early years of school. The framework is based on recent research findings that indicate the importance of students' knowledge of the unit iteration structure. The results of interviews with a sample of 16 Sydney teachers about their understanding of area measurement indicate that they place little importance on structuring in their teaching of measurement. Lessons from the framework were given in four schools and worksamples based on these are discussed.

Introduction

Measurement concepts are introduced in Kindergarten, then developed and extended throughout the primary years. However, students will not learn to estimate and measure successfully if they have inadequate understandings of measurement attributes and processes. They also need to be aware of key differences between measurement and number concepts, in particular, the notions of continuous quantities and variability. Measurement concepts have been found to develop later than number concepts because of the difficulty of dividing a continuous whole into equal sub-units as opposed to counting a set of discrete objects (Piaget, Inhelder, & Szeminska, 1960).

Length concepts are particularly important, as length is usually the first measurement process that students learn about. If students do not have a relational understanding of linear units they are unlikely to succeed in more complex areas of mathematics; for example, area and volume in which length units are extended to two and three dimensions respectively (Battista & Clements, 1996; Outhred & Mitchelmore, 2000).

The importance of understanding the development in the early years has been stressed by Hershkowitz (1990), who says that:

> It is surprising that most of the research that has been done in the last few years has concentrated on 4th to 10th grade (9 to 15 year-olds). It seems as though researchers try to learn about processes that start very early by investigating their traces at more advanced stages. (p. 93)

She listed the following stages through which meaningful learning of measurement develops: (a) conservation of the quantity being measured; (b) the meaning of the measurement unit and unit iteration (arbitrary units, standard units, correct use of measurement tools); and, (c) formulae for calculating the quantity being measured. However, some evidence suggests that the first two aspects might be mutually supportive and that it is not necessary to have a knowledge of conservation before beginning measurement activities (Nunes, Light, and Mason, 1993). Although Hershkowitz lists units and iteration, teachers seem to have paid little attention to the structure of the unit iteration.

The aim of this paper is threefold:

- to propose a framework for teaching measurement of length, area and volume that is based on theoretical research and that emphasises the structure of iterated units;
- to provide information from interviews with teachers about their teaching of area measurement;
- to provide some worksamples of students involved in initial classroom evaluations of the Framework.

Measurement Units and the Structure of the Iteration

Although length, area and volume measurement is taught in primary school there is a considerable body of evidence showing many secondary students do not have a thorough knowledge of these concepts (Carpenter, Lindquist, Brown, Kouba, Silver, & Swafford, 1988; Hart, 1993; Hart, Johnson, Brown, Dickson, & Clarkson, 1989; Schwartz, 1995). More recently, results of the international comparisons of mathematics (TIMMS) have indicated that Australian secondary students also have problems with linear measurement (Lokan, Ford, & Greenwood, 1996).

Commonly reported errors in these studies involve inadequate understanding of:

- the attribute being measured;
- what is counted when informal units are used;
- the use of measurement instruments.

Students first learn to compare quantities on the basis of perceptual judgements to obtain a sense of what a length, area, or volume is. Only gradually are perceptual judgements amended by cognitive intervention and fundamental to this shift would seem to be knowledge of measurement units. Concrete materials and practical activities have been widely recommended but their use has not always been found to be effective because students may not grasp the relationship between different forms of representation (Hart, 1987).

For example, research on students' area concepts suggests that the structure of the rectangular array is not intuitively obvious to students (Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred, 1993) and therefore, they may not perceive the structure of the unit iteration. When students make arrays with concrete materials, such as cardboard tiles, the tiles structure the array whereas to draw an array requires knowledge of its structure. Thus, the step from making an array using concrete materials to representing it pictorially may be a very large one. Emphasising the structure of the unit iteration may assist students to link concrete, pictorial and symbolic representations of measurement concepts.

The concepts of a unit and unit iteration are basic to any understanding of measurement and students will not be able to partition a region into equal parts if they cannot anticipate what the partitioned region will look like (Hiebert, 1989). Students who can draw arrays seem to partition the rectangle by constructing composite units based on rows and columns. Therefore, although it is important to be able to visualise the partitioned region it is also important to have some knowledge of how such a region might be constructed. The results of studies by Outhred and Mitchelmore (2000) and Battista et al. (1998) suggest that the structure of a unit iteration of squares is not well understood by students.

Moreover, students who use multiplication to determine the number of elements in the arrays usually draw arrays using some lines, suggesting that the construction of composite units and the delineation of these units as single entities (by representing their boundaries by lines), are in some way related (Outhred, 1993). A strong relationship was found between measuring the sides of a rectangle and drawing an array using lines. A possible explanation is that drawing lines helps the students to connect the tessellation of unit squares that covers the rectangle with the lengths of its sides. Outhred suggests that students do not realise the significance of the side lengths unless they can visualise and draw arrays as an iteration of rows and columns.

Similar findings emphasising the unit iteration structure have been reported in research on students' understanding of volume (Battista & Clements, 1996; Hart et al., 1989). Volume concepts involve a higher level of visualisation than length or area as some units will be hidden by the layer structure. Campbell, Watson and Collis (1990) investigated the strategies that students in Years 2 to 6 used to find the number of cubes in pictured regular rectangular prisms. Nearly half the problems were solved by counting individual cubes and many students counted only visible cubes. Volume comparisons are more complex than area or length ones because of the difficulty in coordinating three dimensions. Children need time to learn how to decompose a three-dimensional array into layers, and to progress from repeated addition to multiplication as a method of enumerating the unit cubes.

Hart (1989) found that a number of factors contributed to the students' difficulties in understanding the formula for volume of a rectangular prism. Some students filled the containers with blocks then counted them. Although teachers referred to centimetres, students may not have realised that they were referring to the edges of the block; for example, some students measured length as cm^3 because they counted blocks.

Introducing measurement instruction with informal units has been suggested as a way of helping students to see the relationship between the 'continuous' nature of length and the 'parts' that may be counted (Wilson & Osborne, 1988). Australian curriculum documents (Australian Education Council, 1990) recommend the introduction of non-standard units of area before standard units and the measurement of irregular shapes before the measurement of regular geometric shapes.

However, the nature of these parts (or units) that are counted may not be obvious, and problems may arise from the use of informal units unless a fundamental aspect of area measurement is made clear—that the region should be covered without overlap or gaps. Moreover, the use of nonstandard units does not emphasise the structure of the iteration. When learning about area, for example, students should first be given tasks that emphasise the relationship between covering rectangular shapes with square units (Bell, Hughes, & Rogers, 1975). These authors found that early use of irregular shapes, with the associated complexity of fractional units, did not enhance students' concept of area.

The structure of the iteration of length, area, and volume units does not seem to be emphasised in textbooks and curriculum documents, yet it would seem to be central to understanding and linking these concepts. Simon and Blume (1994) found that preservice student teachers needed considerable experience before they began to understand the area of a rectangle as a quantification of surface and to see how the linear measures determined the shape and size of the array. If preservice students have difficulty understanding the structure of a tessellation, then they are unlikely to understand the structure of a three-dimensional packing of cubic units.

Methodology

The K-2 Framework for Learning Measurement has been developed for the Curriculum Support Directorate, NSW Department of Education and Training by the authors. It comprises a theoretical framework, activities, and example lesson plans. A draft document has been released to mathematics consultants for evaluation.

The second researcher interviewed teachers as part of a larger study of professional development models. Sixteen teachers (fifteen females and one male) were individually interviewed in the middle of the school year about their teaching of area concepts. Most of them (11) had more than five years teaching experience; three less than 10 years, four between 10 and 20 years, and four more than 20 years. Seven of the teachers were teaching Year 2, five Year 1, two Year 1/2, one K/1, and one Kindergarten. They were asked about lessons taught, the materials they used, what skills and understanding they were trying to develop, what they believed was the best way to teach area, and was knowledge they thought students would need in three years' time.

Lessons from the Framework were trialed in four schools in a low to medium socio-economic area of Sydney. Some examples of student worksamples were collected to show how students solved a selection of the tasks.

A Framework for Early Measurement Concepts

The Framework (see Figure 1) was designed to provide a sequence of similar conceptual levels for length, area, and volume, prior to the introduction of formal units. The three levels are: (1) identification of the attribute; (2) informal measurement; and, (3) the structure of the iterated unit. The third level does not seem to have been specifically included in measurement teaching programs. However, recent research (Outhred and Mitchelmore, 2000; Battista et al., 1998; Bragg & Outhred, 2000) suggest that this step is crucial because it is not understood by students. These levels provide a conceptual sequence for teaching the content of each strand—students are not expected to be at the same level in each strand.

Table 1
The Conceptual Framework for Learning Measurement

	Length	Area	Volume
Identification of attribute			
Early Level 1	Identify objects of equal length.	Identify congruent areas (possibly after cutting).	Identify volumes (fill or pack containers).
Later Level 1	Direct comparison of lengths.	Direct comparison of areas (e.g., by superimposing).	Direct comparison of volumes (e.g., by filling or packing).
Informal Measurement			
Early Level 2	Find how many identical units fit along a line.	Find how many identical units cover an area.	Find how many identical units fill/pack a container.
Later Level 2	Use numbers of units to compare lengths.	Use numbers of units to compare areas.	Use numbers of units to compare volumes.
Structure of repeated units			
Early Level 3	Replicate a given unit to fit along a line.	Replicate a square unit to cover rectangular areas.	Know how cubic units pack into rectangular containers (layers).
Later Level 3	Relate unit size to the number of units.	Relate unit size to the number of units.	Relate unit size to the number of units.

The Framework document includes a range of activities and sample lesson plans designed to help teachers to link their teaching of measurement concepts across length, area, and volume. It provides several different but related activities at each Level involving practical applications of estimation and measurement. Regular shapes are emphasised because we feel it is important that students develop an understanding of the structure of unit covering in area and unit packing in volume.

The design and presentation of the Framework make it easily accessible to classroom teachers of students in Kindergarten to Year 3. Each of the Levels includes a list of key strategies describing skills and understandings which students may demonstrate at that level. The learning experiences in each Level are referenced to the NSW Mathematics K-6 Syllabus, the Outcomes statements and the strategies to be demonstrated at that level.

The suggested experiences in each level include whole class demonstration and discussion lessons, as well as small group and individual activities. Lesson notes illustrate modelled mathematical language and teacher questioning within a structured lesson. The lessons also show how to introduce the attribute, key aspects of the measurement process, and recording and reporting back to the whole class.

Principles on which the Framework is Based

Students need to identify the attribute before they can directly compare quantities or measure it (Level 1). They also need to know that the quantity is unchanged if it is rearranged (conservation); that lengths, areas and volumes can be subdivided into identical parts (units); and that the number of units used gives a measurement of quantity (Level 2). A key understanding of the measurement process is the repetition (or iteration) of units. Unit iteration involves knowledge of repeatedly placing identical units so that there are no overlaps or gaps. For example, accurately aligning units along a length, constructing an array of units to measure the area of a rectangle, or packing or filling a container to find its volume. Another key understanding is that individual units must be equal in size and they can be combined into composite units, for example, a row of tiles, or a layer of blocks. Experience with composite units will help students to learn that fewer large units are needed to measure a quantity, but smaller units give more precision. Composite units can be used in calculations (and linked to early multiplication, eg. four rows of three squares).

Explicit links must be made so students see how the nature of the units changes from one dimension (length) to three dimensions (volume). For example, diagrams in textbooks use spaces and marks or lines to show length units but students may not know that the spaces on a scale represent unit lengths and that the marks delineate the end-points of the unit centimetres. Such knowledge is important if students are to use scales effectively. Similarly, students may not realise that lines on a grid show the edges of the tiles covering the region or be able to interpret two-dimensional diagrams of cubes.

Results and Discussion

The Interviews with Teachers Concerning Area Concepts

The interviews with teachers revealed their knowledge of area concepts they felt it was important to teach. By mid year the 16 teachers had taught a total of 19 lessons on area while two had not taught any area lessons. The content of the lessons they had taught is categorised in Table 2.

The descriptions teachers gave of the lessons they taught indicates that they conceive "area" to be a process of covering and counting, rather than of subdivision of a surface. Such an emphasis may be detrimental when students move away from concrete materials to more abstract representations that require knowledge of how to subdivide a region (area) or a three dimensional space (volume). No teacher mentioned any structural features of a covering (e.g., gaps or overlaps, congruent units, rows or columns). Even when teachers were specifically asked about skills and understandings these features were not mentioned. Most teachers thought students needed to learn that area is a covering (ten teachers)—only three used the term "surface". One teacher mentioned that the same units should be used for comparison and that an appropriate size unit should be considered.

Table 2
Lessons Taught by the Teachers in the First Part of 1999

Open & closed shapes	2	Covering (with 3D objects)	4
Choose shapes and cover	2	Covering (with approximate 2D objects)	4
Covering (unspecified)	3	Other	4

Four teachers listed only two-dimensional units. Three–quarters of the group mentioned three-dimensional units—blocks, flats, centicubes, shoes, pencil tins, books, bodies, rods, unifix. One mentioned counters, a circular unit, while others used either a mix of 2D and 3D units or 3D materials only. The use of three-dimensional materials, especially blocks, may confuse students about the nature of the units and they may not know whether to focus on an edge, a face or the block itself. Even use of two-dimensional units did not seem to be associated with structural features of covering. For example, in one lesson students traced outlines of themselves onto paper, then covered the shape with envelopes.

The Implementation of the Framework in the Classroom

Work samples were obtained for only a limited number of lessons on length, area and volume. The worksamples have been included to illustrate how young students might engage with activities focussing on the attribute being measured and the structural features of the unit iteration. Figure 1(a) and 1(b) illustrate (a) ordering of length from "big" and getting "smolu" (Kindergarten), and (b) unit iteration of linear units (Year 1). The teacher emphasised in (a) that the lengths being compared must have the same base line; and in (b) that the length (a line) is being measured and how this length is subdivided into equal units. Figure 1(c) show s direct comparison of area (Year 2) by cutting and superimposing.

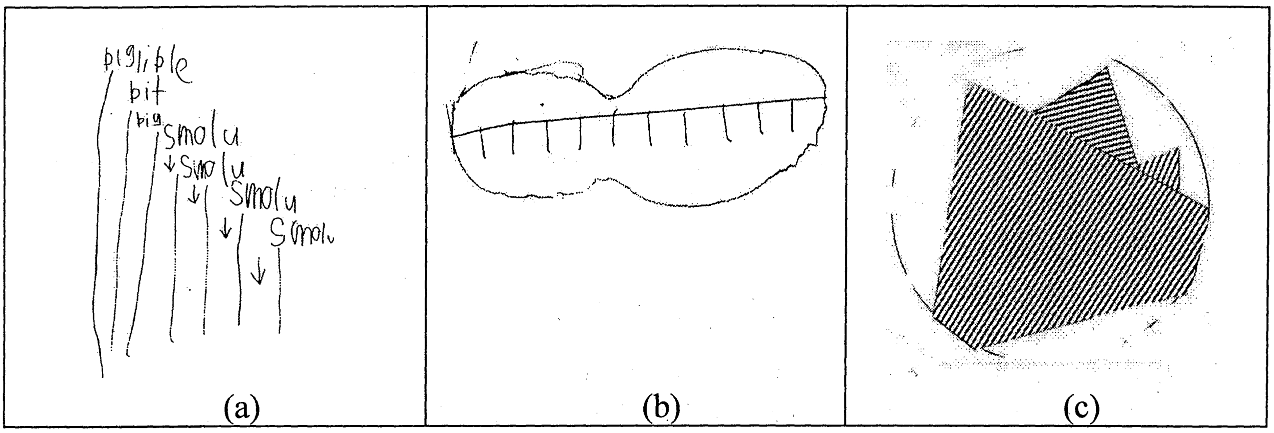
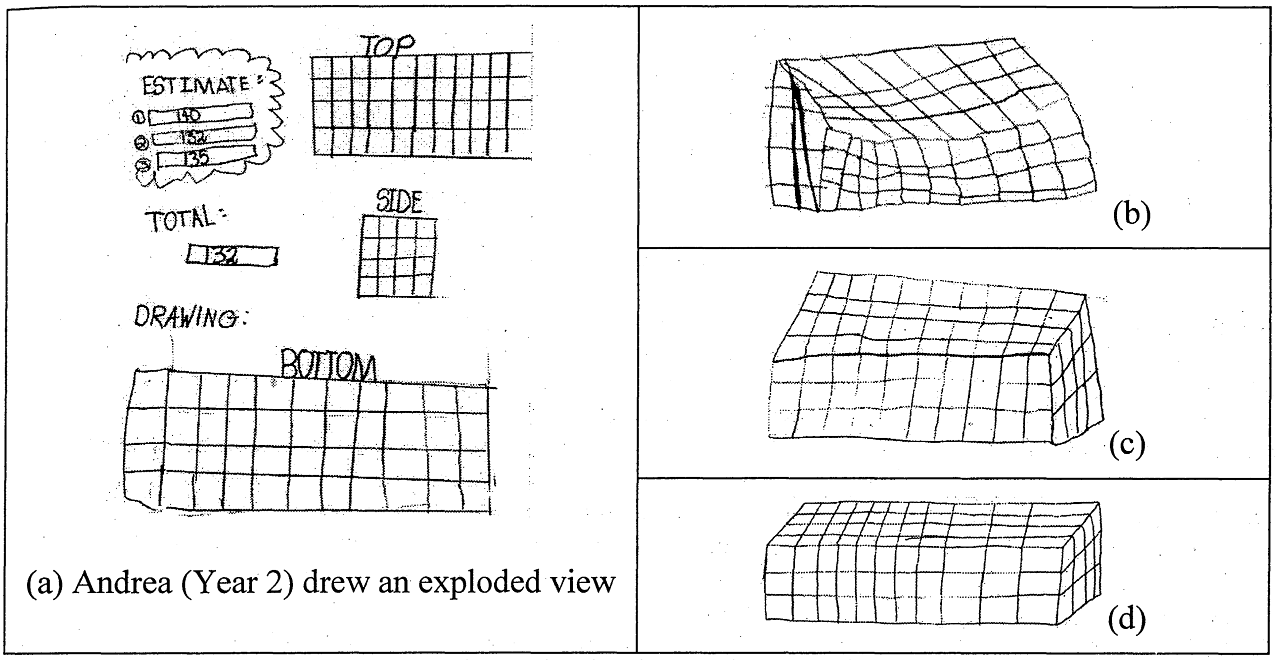


Figure 1. Work samples resulting from length and area activities from the Framework.

Figure 2 shows Year 1 and 2 students' drawings of the structure of the unit iteration for the volume of a rectangular prism. Students packed containers with blocks before making the drawings, which indicate that these young students had a good knowledge of the structural features of volume. Figures 2(a) and 2(d) were drawn by Year 2 students, 2(b) and 2(c) by Year 1 students. These students had previously been involved in a sequence of six lessons on the structural features of an array as part of a study undertaken by the second author giving them a basis on which to build volume concepts.



(a) Andrea (Year 2) drew an exploded view

Figure 2. Work samples showing unit iteration of volume units.

Conclusion

The Framework for Learning Measurement presented in this paper is based on recent theoretical research in measurement. The Framework includes a step that we believe to be crucial to learning measurement—the structure of the unit iteration. We believe that if students do not understand the measurement unit and the structure of the unit iteration they will not be able to progress to symbolic representations of measurement quantities.

The interviews with teachers suggest why students may have little understanding of fundamental properties of measurement units and their structuring in terms of aligning, covering or packing of length, area and volume respectively. Although the sample was a small one, only one of the 16 teachers mentioned features of the measurement unit ("appropriate size and the same units for comparison") and none mentioned the structure of the unit iteration—all seemed to think of area as "covering" something. The use of three-dimensional units may further confuse students about the nature of the attribute they are measuring. Results of a recent study (Bragg & Outhred, 2000) show that, although by Year 5 most students can measure a lines or draw a line of a given length, only high-ability students have a conceptual understanding of linear measurement, that is, the structure of linear units into a scale. Moreover, many students, also in Year 5, when asked what "area" is confused area and volume; for example, "area is the space inside something" or "the number of little cubes in the square" (Bragg, personal communication).

Students' misconceptions may simply reflect teachers' inadequate measurement concepts, combined with syllabus documents that give little direction in terms of key concepts. The worksamples shown in Figures 1 and 2 indicate that when structural features are stressed, students seem to be able to represent these in drawings. The role of such drawings is important as they link the activities with concrete materials to the abstract two-dimensional representations often used by teachers.

The connections between a linear scale, array structure and a three-dimensional packing are important to understanding the development of measurement concepts. These concepts are also closely linked to repeated addition and array multiplication. The difficulty of applying multiplication skills in a meaningful way has been documented for concepts of area and volume (Hart et al., 1989). They found many students continue until well into secondary school to determine both area and volume measures by counting, rather than by multiplying. Such methods do not generalise to fractional dimensions or to the formulae. Knowledge of the measurement unit, the spatial structure of the iteration and its links to multiplication need to be taught to students so teachers need to be made aware of the importance of these aspects of measurement.

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