They can run, but can they hide?

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Secondary and tertiary mathematics teachers often bemoan the fact that many students cannot solve complex questions due to misconceptions involving simple concepts. The authors have earlier reported that the incidence of mathematical misconceptions in both bright and weaker first-year tertiary students was reduced by using a technique based on Piaget's notion of cognitive conflict, and that most of this gain persisted. This study sought to determine whether the same technique diminished the frequency of misconceptions in longer, more complex questions.

Introduction

Teachers of secondary school mathematics or tertiary mathematics will be all too familiar with the following scenario. A student is given a problem to solve; the student presents a solution which would have been correct except for a misconception carried forward from earlier years which destroys the student's attempt to obtain a flawless solution.

The authors, having witnessed events such as this over a long period of time, decided to pursue a possible solution to the problem. The various stages of their research have been reported over the last four years. Their first step was to devise a list of mathematical misconceptions which students exhibit. Davis (1984) had previously considered the different types of mathematical misconceptions. Swedosh (1996) recorded a study which examined the nature and frequency of mathematical misconceptions displayed by students entering tertiary mathematics subjects at two Australian universities.

Having investigated the nature of mathematical misconceptions demonstrated by these tertiary students, the authors considered strategies which might reduce or even eliminate these misconceptions. Tirosh (1990) and Vinner (1990) had earlier supported the conflict teaching approach, based on Piaget's notion of cognitive conflict, as being successful in helping students overcome misconceptions. To put the conflict teaching approach into operation, teachers discuss with the students the inconsistencies in the thinking of the students in order to have the students realise that their conceptions are inadequate and in need of modification. Swedosh and Clark (1997) communicated an experiment designed to help a group of bright first year students (as seen from their high secondary mathematics and university entrance scores) at the University of Melbourne (U. of M.) overcome their mathematical misconceptions so that the success or otherwise of the conflict teaching approach could be determined. The authors found the use of this strategy to reduce mathematical misconceptions to be a remarkable success.

The authors then conjectured that it may be possible that, if these students were tested at some later time, some of them may have reverted to their (often long-held) misconceptions. Swedosh and Clark (1998) described an investigation into whether the reduction in mathematical misconceptions exhibited by the group of bright first year students at the U. of M. referred to in the previous paragraph persisted or whether the effect was temporary. The authors' research showed that the improvement resulting from the use of the conflict teaching approach was much more than a short term quick fix, with the students showing that substantial benefits were still evident one year after the initial reduction in mathematical misconceptions had occurred.

The strategy used to this point, based on Piaget's notion of cognitive conflict, was clearly very useful in reducing mathematical misconceptions with the group of bright first year students with whom the investigation was carried out. The question now arose: for students who are less able mathematically, would such an improvement occur? The reason for asking this question was two-fold: The bright students had been found to be capable of recognising the inconsistencies in their previous thinking and then learning the correct concepts quickly, and they had indicated during their discussions with the authors that they were embarrassed by making the errors that they did and they had a strong desire to remedy the situation. The authors were of the view that both of these qualities might be more pronounced with bright students than with less able students. Swedosh (1999) related an investigation to determine the effectiveness of the conflict teaching approach with a group of less able students; the experiment provided strong evidence that a teaching strategy based on Piaget's notion of cognitive conflict can be successfully employed to significantly reduce the frequency of mathematical misconceptions exhibited by less able students.

Having now shown the efficacy of the conflict teaching approach in reducing mathematical misconceptions in students of different ability levels, and that the benefit persists over time, the authors were able to assert that

> The major benefit to be gained from incorporating this strategy, which is extremely simple to implement, into one's teaching, is that not only is it likely that fewer students will have these misconceptions, but, as a result of this, many will directly improve their chances of being successful in their future studies of mathematics (Swedosh, 1999, p. 476).

The investigation reported in this paper considered the problem as to whether a mathematical misconception which had been eliminated or at least vastly reduced in a particular group of students might nevertheless be exhibited if the concept was embedded in some longer, more complex question. For example, consider the question: "Solve for x: $x^2 = 4x$ for which a common misconception is that $x = 4$ is the only solution. Suppose that this misconception has been eliminated, and that students respond $x = 0$ or $x = 4$. Does the misconception re-occur if students are asked a question such as: "Use the fact that $\ddot{x} = \frac{d\dot{x}}{dt}$ to solve the differential equation $\ddot{x} = \dot{x}$"?

Methodology

The methodology was essentially the same as that reported in Swedosh and Clark (1997) with a few small differences. A short test was administered to a group of first year students who were studying Applied Mathematics at the U. of M. in Second Semester, 1999. In the earlier study, a test had been used which was made up of questions similar to those posed in earlier tests at the U. of M. and at LaTrobe University (Worley, 1993) and to those which appeared in the list of misconceptions provided in "Algebraic Atrocities" (Margulies, 1993, p. 41). Each of the questions had previously yielded a high frequency of misconceptions (Swedosh, 1996) so that if students did possess a particular misconception, the question would provide them with the opportunity of exhibiting that misconception. The tests compiled for this study used some of the questions from the earlier study, but also comprised questions in which the concepts being tested in the simpler questions were embedded in more difficult problems so that the concept being tested was not so obvious. Students had no prior warning that there would be any tests.

Questions on all test papers were examined carefully to gain information on the misconceptions exhibited and the nature of each response was recorded. That is, whether the response could be categorised as correct, a misconception, or another wrong answer. The test was administered in the third week of the semester.

Approximately twenty minutes was then spent during a lecture in the fifth week of semester using the conflict teaching approach in an attempt to decrease or eliminate the frequency of misconceptions exhibited on the first test and replace them with the correct concept. For example, if students were to exhibit the misconception that if we were to solve x: $x^2 = 4x$ and answer that $x = 4$ was the only solution, suitable approaches would be to demonstrate by substitution that $x = 0$ is also a correct solution or to discuss the fact that we are dealing with a quadratic and expect two solutions. Reference to the relevant parabola could also be used. At this point, the student (hopefully) would be able to see that the misconception leads to an incomplete and therefore incorrect answer, and would be happy to discard it. The correct concept would then be taught (putting the quadratic on one side of the equation so that it equals zero, factorising, and hence finding both solutions).

The six questions on the first test were used again in the second test but an extra question was added. This was due to a situation which arose in an assignment question in the middle of the semester which the authors thought needed further consideration in that a high proportion of students made the same misconception. The misconception was to solve an equation for a variable squared, and then give the answer as the positive root. In this question, the initial conditions ensure that the correct answer is the negative root and students were expected to consider both roots, and then choose the positive one. The second test was administered to the same class in the tenth week of semester (five weeks after the teaching session previously described). The five week break was important so that students were answering questions on the second test based on their knowledge of the relevant concepts rather than the memory of what they had recently been taught. In the case of Question 7, a teaching session was carried out in week 12.

The frequency of occurrence of misconceptions on each of the questions was then calculated, and these were compared with the frequencies exhibited on the same questions before the teaching session. In the case of the new (assignment inspired) question, the frequency of misconceptions was compared with the frequency exhibited on a similar question on the final exam.

The Sample

The test was administered to students enrolled in first year Applied Mathematics in Semester 2, 1999 at the U. of M. All students in this class had successfully completed a fairly demanding prerequisite mathematics subject in Semester 1 which only about 25% of first year students were permitted to attempt. So that a meaningful comparison of the results could be made, and to ascertain whether the intervention of the teaching approach had caused a reduction in the frequency of misconceptions, a sample of these students who have homogeneous backgrounds are reported on. The students reported on in this study sat both the first test and the second test, and had completed Specialist Mathematics 3/4 (SM) as part of their Victorian Certificate of Education (V.C.E.). There were 57 students in this category.

In Specialist Mathematics 3/4 there are three Common Assessment Tasks (CATs) and students are awarded a grade from E to A+ corresponding to a ten point scale from one to ten. Using this scale, the 57 students in this sample had an average mark for SM CAT 1 (the challenging problem) of 9.91, an average mark for SM CAT 2 (facts and skills) of 9.47, and an average mark for SM CAT 3 (the analysis task) of 9.26. They had an average Equivalent National Tertiary Entrance Ranking, or ENTER (a percentile with the highest possible ranking of 99.95 and with about 23 students for each 0.05 -- 0.05% of students in the state is about 23), of 97.64. Clearly this was an extremely able group of students, even more able than the group which had been considered in a previous study (Swedosh and Clark, 1997) despite the fact that that group was extremely good. In the previous study, the average TER was 93.56. The relationship between the TER, and the new measurement, the ENTER, is that participation rates are now taken into account in the calculation. This has the effect of elevating the rank, in this case by about two points. Those excluded from this study include mature age students as well as interstate and overseas students whose backgrounds were quite different to the 57 students considered herein.

The Tests

The first test consisted of six short answer questions. The first three questions were relatively straight-forward, whereas the latter three were more complex but had the same concepts embedded. The second test contained these same six questions plus one extra question which is numbered 7 below. The exam question is shown as E below. As previously stated, each question "invited" students to make a particular misconception, should that misconception exist. Students were given twenty minutes to complete the test. An analysis was conducted on each question and a count was taken of how many students had the correct answer, how many had exhibited the misconception, how many had another wrong answer, and how many had not attempted the question.

The eight questions used are shown below, and the most common misconception(s) are shown to the right of each question:

1. Solve for x: $x^2 = x$. — $x=1$

2. Given that $\sin\frac{\pi}{6}=\frac{1}{2}$, evaluate $\sin\frac{7\pi}{6}$. — $\frac{1}{2}$, $\frac{7}{2}$

3. Indicate whether the statement is True or False. (circle one)

   $k\left(50-\frac{x}{5}\right)(80-2x)=k(250-x)(40-x)$ T / F — True

4. Use the fact that $\ddot{x}=\frac{d\dot{x}}{dt}$ to solve the differential equation $\ddot{x}=2\dot{x}$. — $\dot{x}=2x+c$

5. When preparing to sketch a polar graph, you are required to first complete a table of values. Given that when $\alpha=\frac{\pi}{3}$, $\cos\alpha=\frac{1}{2}$, fill in the three blanks in the following table for the relation $r=2\cos\theta$. DO NOT SKETCH THE GRAPH. — 2, 3, 4; $\frac{1}{2}$, 1, $\frac{1}{2}$

θ	α	2α	3α	4α
r	1			

6. The differential equation governing a particular chemical reaction is
$\frac{dx}{dt} = k(40 - \frac{2}{5}x)(50 - \frac{1}{5}x)$.

Can the solution to this differential equation be expressed in the form
$\log\frac{100 - x}{250 - x} = kt + c$?

Yes

7. Solve the differential equation $\frac{dv}{dt} = \frac{1}{2v}$ subject to $v(0) = -2$. $v = \sqrt{t+4}$

(Your answer should explicitly express v as a function of t.)

E. Solve the differential equation $2xy\frac{dy}{dx} = x+1, x > 0, y(1) = -2$ $y = \sqrt{x + \log x + 3}$

(Your answer should explicitly express y as a function of x.)

Teaching Method

The teaching took place two weeks after the first test. As specified in Swedosh and Clark (1997),

> The method essentially involved showing examples for which the misconception could be seen to lead to a ridiculous conclusion, and, having established a conflict in the minds of the students, the correct concept was taught (p. 476).

When the correct concepts were taught, care was taken to use slightly different examples to the questions on the tests to ensure that students had to correctly use the concept, not just remember what they had been shown. Some examples of what was taught are shown below.

$x^2 = 9x,\ x^2 - 9x = 0,\ x(x-9) = 0,\ x = 0, 9$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$, as $\frac{4\pi}{3}$ is in the third quadrant.

$k\left(10 - \frac{x}{2}\right)(200 - 5x) = k(20 - x)(40 - x)$, (True or False?)

$k\left(\frac{1}{2}\right)(20 - x)(5)(40 - x) = k(20 - x)(40 - x)$; $\frac{5k}{2} = k$ which is only true for $k = 0$.

$$\frac{dv}{dt} = \frac{1}{4v},\ v(0) = -3;\ \int v\,dv = \int\frac{1}{4}\,dt;\ \frac{1}{2}v^2 = \frac{1}{4}t + k;\ v(0) = -3,\ k = \frac{9}{2};\ v^2 = \frac{1}{2}t + 9;\ v = \pm\sqrt{\frac{1}{2}t + 9}.$$

But only the negative root satisfies the initial conditions, so $v = -\sqrt{\frac{1}{2}t + 9}$.

Results

The tables below summarise the responses given by the 57 students to the eight questions shown earlier. It is important to note that in the tables below, "Question 7 Before" refers to the first time this question was attempted (on Test 2) and "Question 7 After" refers to the second time this question was attempted (on the exam). Table 1 shows the frequencies of each category of response given before and after the respective teaching session. Table 2 shows the proportions of each response before and after teaching session for those who attempted the question. In each table, 'Q' is the question number.

Table 1
Response Frequencies Before and After Teaching

Q	Correct		Misconception		Other wrong		No attempt	
	Before	After	Before	After	Before	After	Before	After
1	53	56	4	1	0	0	0	0
2	57	57	0	0	0	0	0	0
3	54	55	3	1	0	0	0	1
4	9	24	47	31	1	0	0	2
5	42	45	7	4	4	6	4	2
6	18	42	12	13	7	0	20	2
7	29	31	24	17	3	9	1	0

Table 2
Response Percentages for those Attempting the Question

Q	% Correct		% Misconception		% Other error	
	Before	After	Before	After	Before	After
1	93.0	98.2	7.0	1.8	0.0	0.0
2	100.0	100.0	0.0	0.0	0.0	0.0
3	94.7	98.2	5.3	1.8	0.0	0.0
4	15.8	43.6	82.5	56.4	1.8	0.0
5	79.2	81.8	13.2	7.3	7.5	10.9
6	48.6	76.4	32.4	23.6	18.9	0.0
7	51.8	54.4	42.9	29.8	5.4	15.8

It can be seen from Table 2 that for every question except Question 2 (where it was impossible), the proportion of misconceptions exhibited decreased. Question 6 actually had one more student exhibit a misconception after the treatment but there were ten times as many students not able, or not willing, to make an attempt on the first test. Many of the gains were not large but many could not be due to the very high baseline set by this group. This was particularly evident on the straight-forward questions. The reason that the "Other error" proportion was high on the exam (Question 7 After) may be due to exam pressure.

Table 3 shows the pattern of student responses from the first test to the second test. CC means that the student gave the correct answer on both tests. CM means that the student gave the correct answer on the first test and exhibited a misconception on the second test. MC means that the student first exhibited a misconception and then gave the correct answer. M M means that the student exhibited a misconception on both tests. Only responses categorised as correct answers or misconceptions are considered in this table.

Table 3

Pattern of Student Responses

Q	CC	CM	MC	MM
1	53	1	3	0
2	57	0	0	0
3	54	0	1	1
4	6	3	17	28
5	34	3	4	1
6	15	3	10	3
7	24	2	5	14

For all questions (except Question 2 again), there were more students in the MC category than in the CM category. Questions 4 and 6 stand out as having had large improvements. The data in Table 3 can be analysed using McNemar's test for correlated proportions which focuses on discordant pairs (CM and MC). This test gives a very significant result for the Question 4 data ($p = 0.003$). The data for Question 6 is significant for a one-sided test ($p = 0.046$) but just fails to be significant with a two-sided test ($p = 0.092$). It is a little disturbing that there are a small number of students who gave a correct answer the first time and a misconception the second. The authors believe that it is likely that many of these students do not have a real understanding of the concept and that the answer given might therefore vary arbitrarily and that they "fluked" the correct answer on the first occasion. In any case, the outcome for those students was not improved.

Conclusions

The students considered in this study were so proficient at correctly answering simple questions involving basic concepts that little improvement could be made on such questions. On the more complex questions in which the same concepts were embedded, there was an improvement in all questions and a substantial improvement in two of the four. It seems that as we expected, it is more difficult for students to recognise exactly what is required and which concepts are involved in more complex questions. It appears that they become so immersed in the larger question, they lose track of the detail required or lose concentration regarding the "nuts and bolts" of the question.

The results indicate that while it is more difficult to reduce the rate of misconceptions when concepts are embedded in more complex questions, there is a clear improvement in student performance on these questions after the conflict teaching approach has been used. While not as dramatic as the improvement found in previous studies on simpler questions, the benefits are still considerable. This method has been shown to be useful in combating the very serious problem of students not being able to solve a non-trivial question without being hamstrung by a lack of mastery of some basic concepts along the way.

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[1] *Acknowledgements*. We would like to acknowledge the advice of Robert Maillardet and Ian Gordon regarding the statistical analysis. Their efforts are sincerely appreciated.