Integration and Compensation in Accurate Mental Computation

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Literature at national and international levels argues the importance of including mental computation in a mathematics curriculum that promotes number sense. Characteristics of good mental computers have been documented. However, there is conflicting evidence about what constitutes high performance in mental computation – high accuracy as a result of efficient mental strategies and number sense or high accuracy without accompanying number sense. The purpose of this paper is to present findings of a study that investigated accurate mental computers and the factors that supported accuracy.

Mental computation is defined as “the process of carrying out arithmetic calculations without the aid of external devices” (Sowder, 1988, p. 182). Literature at national and international levels argues the importance of including mental computation in a mathematics curriculum that promotes number sense (e.g., Klein & Beishuizen, 1994; McIntosh, 1998; Reys, Reys, Nohda, & Emori, 1995). It has been posited that when children are encouraged to formulate their own mental computation strategies, they learn how numbers work, gain a richer experience in dealing with numbers, develop number sense, and develop confidence in their ability to make sense of number operations (Kamii & Dominick, 1998; Reys & Barger, 1994).

Anghileri (1999) claimed that mental computation was calculating *with* the head, rather than merely, *in* the head, that is, mental computation is calculating using strategies with understanding. However, research has found that students can compute accurately *in* the head without understanding (e.g., McIntosh & Dole, 2000; Heirdsfield, 1996); that is, high performance in mental computation can be achieved without accompanying number sense.

While some research appears to indicate that accuracy in mental computation is a result of efficient mental strategies, other research has reported accuracy as a result of employment of strategies that reflect pen and paper algorithms. The purpose of this paper is to provide evidence for supporting factors for both types of accuracy, by presenting the findings of a study that investigated addition and subtraction mental computation in seven and eight year old students, and by formulating frameworks that explain these differences. In particular, a fundamental aim was to identify factors and the relationship between factors, which influence children’s addition and subtraction mental computation strategies. In order to commence this study, the literature was consulted to identify some possible factors on which to base the initial investigation. The literature, presented here, drew from both literature referring directly to mental computation and other literature referring more generally to mathematics.

Research on mental computation has proposed specific connections among mental computation and aspects of number sense, in particular, number facts and estimation (e.g., Heirdsfield, 1996). Other research relating to computation (in particular, children’s natural strategies) has reported connections with numeration (e.g., place value) and effects of operation on number (e.g., Kamii, Lewis, & Jones, 1991).

Relationships have been posited between mental computation and affects (e.g., Van der Heijden, 1994), where affects include beliefs (with respect to mathematics, self, teaching, and social context), attitudes (including self efficacy and attribution) and emotions (McLeod, 1992). Beliefs about the nature of mathematics can be manifested in a student's disposition – mastery orientation or performance orientation (Prawat, 1989). In relation to computation, mastery oriented students would aim for understanding and flexibility. Here, monitoring, checking, and planning might be evident. Whereas, performance oriented students would tend to aim to complete a task as quickly as possible, and not attend to understanding and reflection.

Proficient mental computers are considered to be flexible in their choice of strategies (e.g., Sowder, 1994). Such effortful, reflective and self-regulatory behaviour should involve metacognition. Metacognition can be considered to have three components: *metacognitive knowledge* (knowledge of own thinking), *metacognitive strategies* (planning, monitoring, regulating and evaluating), and *metacognitive beliefs* (perception of own abilities and perception of a particular domain) (Paris & Winograd, 1990).

Memory and the part it plays in mental computation have also been discussed in the literature. Hope (1985) argued that superior short-term memory was not necessary for proficient mental computation; rather, interest, practice, and knowledge were more important factors. Heirdsfield (1999) found that a superior short-term memory was unnecessary for a student who was accurate and flexible with mental addition and subtraction. However, Heirdsfield also found that the *mental image of pen and paper algorithm* strategy always used by accurate/inflexible students tended to place heavy demand on short-term memory. A model for working memory consisting of several parts (eg, Baddeley, 1986, 1990, 1992; Baddeley & Logie, 1992; Logie, 1995) was used as a basis for further investigation of mental computation. In brief, the *central executive* component provided a processing function and a co-ordinating function, which included information organisation, reasoning, retrieval from long-term memory (access), and allocation of attention. The *phonological loop* (PL) was responsible for storage and manipulation of phonemic information, for instance, rehearsal of interim calculations. The *visuospatial scratchpad* (VSSP) dealt with holding and manipulating visuospatial information. This may involve representation of numbers in the head, or positional information of algorithms.

In summary, research on mental computation and number has proposed connections among mental addition and subtraction, number sense (e.g., number facts, estimation, numeration, effects of operations on number), affective factors (including beliefs, attributions, self efficacy, and social context in classroom and home); and metacognitive processes. Further, it appeared that memory might have an effect on mental computation.

The Study

The research consisted of a pilot study and a main study. Both studies were based on interviews developed to investigate mental computation and other aspects that were identified from the literature. The findings of the pilot study informed the main study.

Subjects

The subjects were Year 3 students from two Brisbane Independent School that serve high and middle socio-economic areas. The students were selected (from a population of three Year 3 classes, 60 students in all) after participating in a structured mental computation selection interview. As proficiency in mental computation was defined in terms of both flexibility and accuracy, both these factors were considered when selecting the students. As a result of their performance on the selection items, students were identified as accurate and flexible (4 students) and accurate and not flexible (2 students).

Instruments

The students participated in indepth interviews, which addressed mental computation strategies, number facts, computational estimation, numeration, number and operations, and investigated metacognition and affect. These tasks have been described elsewhere (Heirdsfield & Cooper, 1997). As a result of analysing the pilot study, another factor, memory seemed to impact on mental computation, particularly for the accurate and inflexible student. Therefore, memory tasks were also presented to the students. These addressed short-term recall, short-term retention, and executive planning.

Interview Procedures

The students were withdrawn from class and participated in a series of videotaped semi-structured clinical interviews in a quiet room in the school.

Analysis

For the purposes of identifying flexibility in mental computation, mental computation strategies were identified using the categorisation scheme (based on Beishuizen, 1993; Cooper, Heirdsfield, & Irons, 1996; Reys, Reys, Nohda, & Emori, 1995; Thompson & Smith, 1999) that divided the strategies into the following categories: (1) *separated* (e.g., 38+17: 30+10=40, 8+7 = 15 = 10+5, 40+10+5 = 55); (2) *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55); (3) *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55); and (4) *mental image of pen and paper algorithm* – following an image of the formal setting out of the written algorithm (taught to almost automaticity in the schools the students attended). .

Mental computation responses were analysed for strategy choice, flexibility, accuracy, and understanding of the effects of operation on number, numeration, computational estimation, and number facts. Analysis of the interviews investigating these individual factors was also undertaken, with the intention of exploring connections with mental computation. Students' responses were also analysed for metacognition and affects. For the memory tasks, scores and strategies were recorded.

Each student's results for aspects of number sense, metacognition, affects and memory were summarised. These summaries were combined for each of the computation types: accurate and flexible, and accurate and inflexible, so that comparisons could be made between the two types of accurate mental computers, and frameworks formulated to explain the differences between the two.

Results

Firstly, the accurate and flexible mental computer employed a variety of efficient mental strategies (including *wholistic* and *aggregation*). In contrast, accurate and inflexible computers employed *mental image of pen and paper algorithm* throughout the mental computation interviews.

Proficient Mental Computers

Proficient mental computers possessed well-integrated knowledge bases (see Figure 1). They were fast and accurate with their number facts, and used efficient number facts strategies when facts were not known by *recall*. Also, number facts strategies were extended to efficient mental computation strategies. Good numeration understanding (particularly canonical, noncanonical, multiplicative, and proximity of number) and some number and operation supported efficient 'alternative' strategies. Further, numeration understanding (particularly proximity of number) and understanding of the effect of operation on number appeared to be essential for employment of the mental computation strategy, *wholistic* (e.g., 246+99=246+100-1).

Proficient students had accurate perceptions of their ability to solve the mental computation tasks, and they used metacognitive strategies (e.g., monitoring, reflecting, regulating, and evaluation).

Beliefs in self seemed to be associated with a belief about the place of the teacher in the student's learning; for instance, confidence in self-initiated strategies (c.f., teacher-taught strategies) supported flexibility in mental computation. Although there was not always evidence of the belief that mathematics makes sense, that belief was strong in one student.

Reasonable levels of short-term memory and central executive were required for both accuracy and flexibility. The central executive would have supported efficient processing and coordinating of proficient mental strategies. With regard to slave systems in working memory (phonological loop (PL) and visuospatial scratchpad (VSSP)), it was evident that the phonological loop supported retrieval of number facts from LTM (long term memory), and holding and rehearsal of interim calculations. However, it is posited, there was not such a demand on the PL for rehearsing interim calculations as efficient mental strategies resulted in fewer interim calculations that needed to be rehearsed. There was further evidence of the functioning of PL by the results of the Digit Span Test. However, there was little evidence of the use of the VSSP, except in one case where a student stated that he "saw" MAB in his head when calculating. It would be expected that numbers would be represented in some visual form, yet no student reported this. Certainly, the students did not appear to be manipulating symbols in their head, but they did not report "seeing" numbers in any form.

There were other factors that did not contribute directly to proficiency, but were sometimes present at a threshold level (base degree of knowledge proficiency). If these factors were present, they had positive effects. Most of the students exhibited some estimation understanding, although in only one instance, did estimation appear to support mental computation.

In summary, proficiency (accuracy and flexibility) in mental computation was supported by a rich network of cognitive, metacognitive and affective components. Further, there was complexity of factors contributing to these components.

Accurate and Inflexible Computers

The students who were accurate, but did not use alternative and efficient mental strategies, were similar to students who were both accurate and flexible in only a few factors. Although only one student fitted this category clearly, factors that distinguished the two students who were categorised in this group were taken into account when formulating the framework for accurate and inflexible mental computers. Comparing the accurate and inflexible mental computers with the proficient mental computers, it was evident that these students had more limited and less connected knowledge bases (see Figure 2).

Numeration was evident at a threshold level, rather than being an essential component of the students' mental computation. Further, some aspects of numeration were not evident.

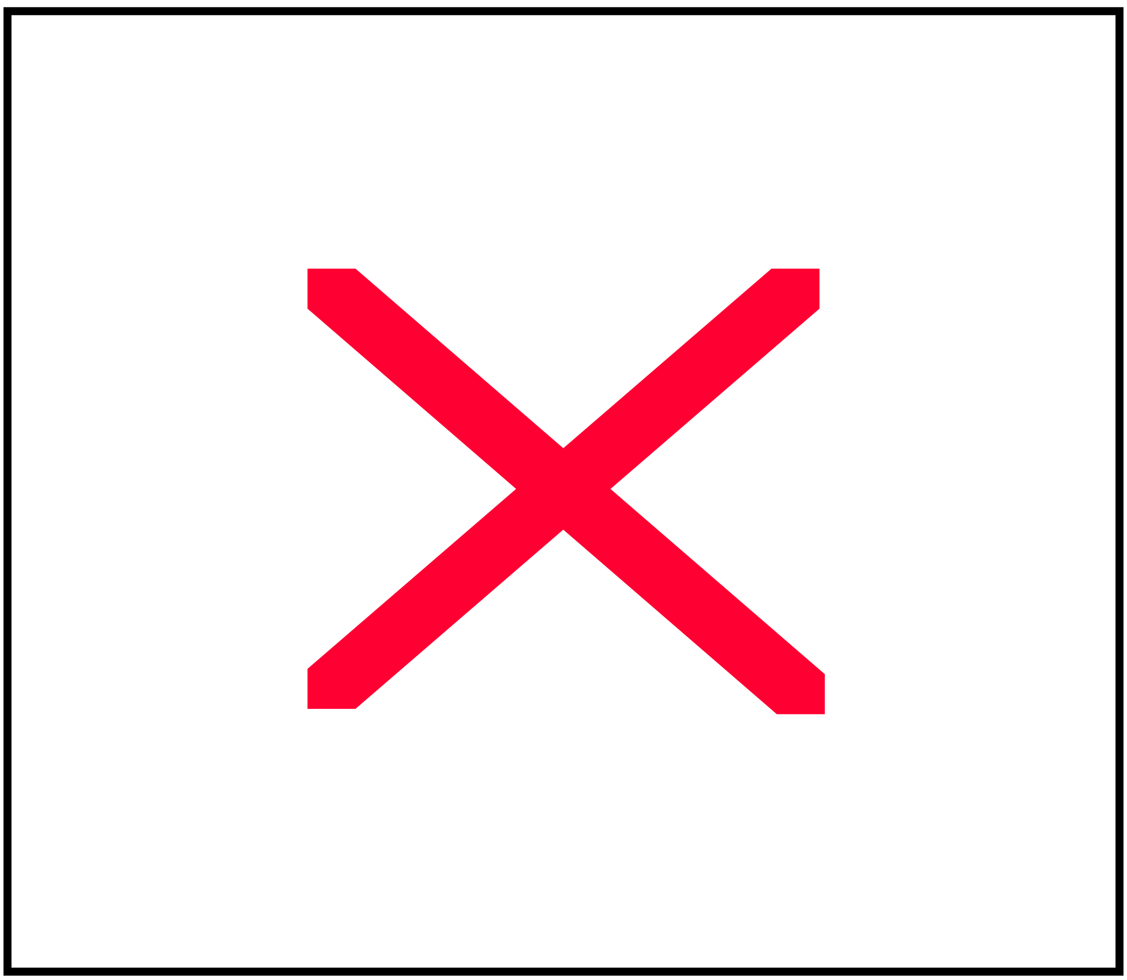

Figure 1. Framework for proficient mental computation

It was evident that fast and accurate number facts supported accuracy in mental computation. However, number facts *strategies* did not seem to be important, as one of these students used *count* as a backup strategy in both the number facts test and for interim calculations in mental computation. Although the other student possessed some efficient *DFS*, she did not employ them in mental computation. Instead, she also resorted to *count* when number facts could not be recalled.

Numeration was not well understood, however, there did seem to be some threshold knowledge of canonical and noncanonical numeration understanding. Canonical and noncanonical understanding contributes to successful employment of pen and paper algorithms. Knowledge of multiplicativity and proximity of number was not evident. It is posited that the absence of knowledge of proximity of number and knowledge of the effect of operation on number resulted in students' not employing the mental strategy *wholistic*. One of these students exhibited more numeration understanding than the other, and it is posited that this understanding supported the alternative strategies she used later in the interviews. Estimation did not support these students' mental computation. However, when applying pen and paper algorithms, a sense of the answer might not be perceived as being important, as the student calculates from right to left, and treats the components as digits, rather than in their place values.

Strong beliefs in teacher-taught strategies and teacher feedback, and attributing success to a teacher contributed towards selection of the teacher-taught strategies for mental computation. Self-efficacy might have supported accuracy.

Although these students did not demonstrate metacognitive strategies, they did hold accurate perceptions of their ability to perform the tasks. Metacognitive beliefs might have supported accuracy.

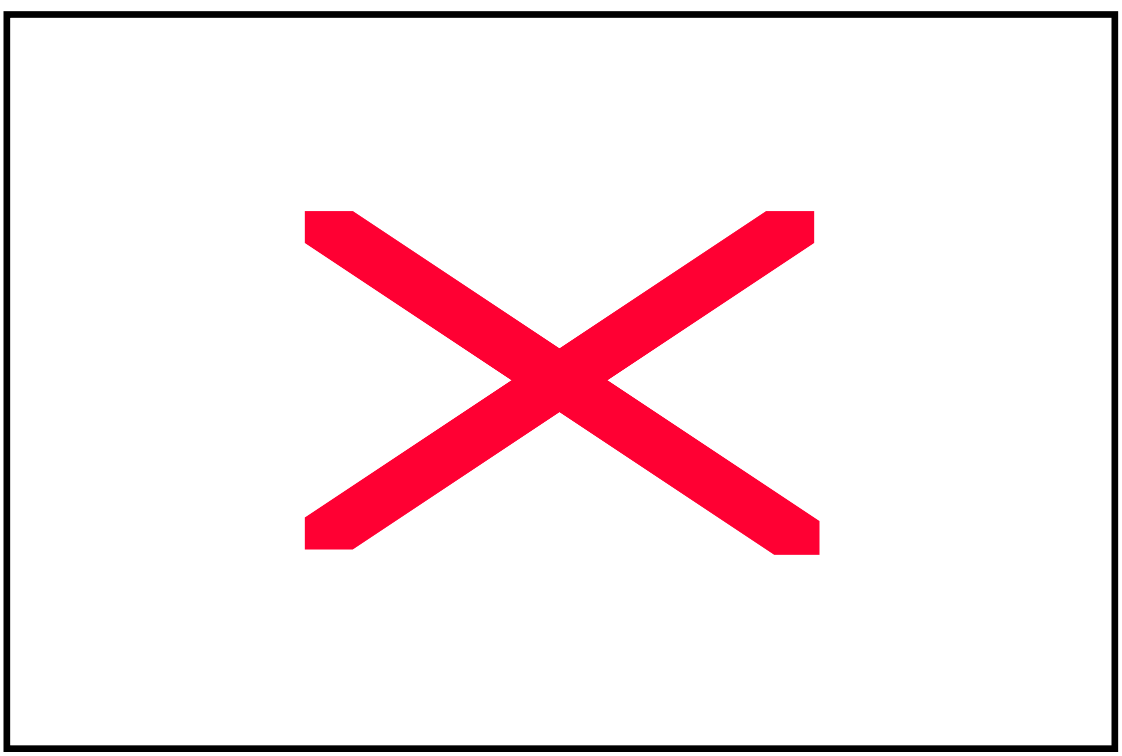

Figure 2. Framework for accurate and inflexible mental computation.

Reasonable levels of short-term memory and executive functioning were required for accuracy. However, there might not have been such a demand for the central executive as automatic strategies were retrieved from LTM. With regards to slave systems in working memory, it was evident that the phonological loop (PL) supported retrieval of number facts from LTM, and holding and rehearsal of interim calculations (of which there were many). There was further evidence of the functioning of PL by the results of the Digit Span Test. Further, there was evidence that the visuospatial scratchpad (VSSP) supported these students' mental strategies. The visual representation of the pen and paper algorithm, including interim calculations was stored and manipulated in the VSSP.

In summary, accuracy and inflexibility in mental computation resulted from a less extensive knowledge base than that possessed proficient mental computers, and certain affective and metacognitive factors. Beliefs about self and teaching directly contributed to the selection of teacher-taught strategies.

Conclusions

Proficient mental computers chose alternative and efficient strategies, as they possessed extensive and connected knowledge bases to support these strategies. In contrast, accurate and inflexible students did not "choose" a strategy; rather they simply applied an automatic strategy (teacher-taught). Although supporting knowledge was limited, it was sufficient for implementation of a well-rehearsed algorithm. Thus, although both groups of students were accurate, considering the amount of time that is spent in teaching the written procedures, it appears that time could be better spent in having students develop their own strategies.

There was evidence of the importance of connected knowledge, including domain specific knowledge, and metacognitive strategies for proficient mental computation. This demonstrated the need for teaching practices to focus on the development of an extensive and integrated knowledge base to develop understanding; that is, concepts, facts, and strategies should not be learnt in isolation.

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