The Influence of Visual Representations on Mathematical Problem Solving and Numeracy Performance

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The purpose of this study was to examine relationships between different forms of problem representation and students' performance on a number of mathematics measures including problem solving, spatial ability and numeracy sense. Participants (58 Grade 6 students) solved mathematics problems and reported their solutions. The participants mode of representation, which included problem-solving success and representational preference, was placed along a visual-nonvisual continuum. The mode of representation did not influence problem solving or numeracy performance. However, when students at the extremes of the continuum were investigated, students who predominantly used visual methods performed better on these measures.

Both psychologists and education researchers interested in imagery have classified problem solvers as belonging to one of three categories: (a) *visualizers*, who had a preference for holistic approaches involving extensive use of visual methods; (b) *non-visualizers or verbalizers*, who had a preference for more verbal approaches; and (c) those who tended to use *both* visual and non-visual methods (Krutetskii, 1976; Lean & Clements, 1981; Lowrie & Kay, in press; Presmeg, 1985, 1986; Suwarsono, 1982).

Suwarsono (1982) developed a Mathematical Processing Instrument (MPI) that was used to assess a student's preference for solving mathematical problems in either a visual or nonvisual mode. He claimed that the MPI could locate both problem solvers *and* problems on a single unidimensional verbalizer-visualizer scale with the instrument being used extensively in the research literature (including Hegarty & Kozhevnikov, 1999; Lean & Clements, 1981; Lowrie & Kay, in press). In all of these studies a participant was classified along the continuum based on the type of representation predominantly used to solve the problem. This was the case whether or not the problem was solved correctly. Recently, other writers have suggested that categorizing students along a visualizer-verbalizer continuum was too general a classification when considering problem representation (Hegarty and Kozhevnikov, 1999; Riding & Read, 1996). In the present study the visualizer-verbalizer classification takes into consideration both the manner in which the problem is represented and the actual success in solving.

The Role of Imagery in Problem Solving

It is evident that visual imagery plays a vital role in the processing of information (Bishop, 1989; Del Grande, 1990; Dreyfus, 1991; Presmeg, 1986; 1992). Recent problem-solving models (including those of Goldin, 1987; Lowrie & Hill, 1996; Pirie & Kieren, 1991; 1992) have emphasized the role imagery plays in the processing of information. For example, Pirie and Kieren (1992) related problem-solving processes to conceptual development. They argued that the problem solver goes back from a higher level of processing to a richer visual level when necessary. This is consistent with Kaufmann's (1979, 1990) notion that imagery is particularly useful where the need for processing is high, such as occurs under task-novelty conditions. In such circumstances imagery can provide a backup system that facilitates access to a set of simpler cognitive processes.

It has been suggested that some people have a preference for using “visual” methods when attempting to solve problems while others prefer “verbal” or “analytic” approaches. Riding and Pearson (1994), for example, have argued that an individual’s cognitive style would often determine the manner in which a task was represented. Studies by mathematics educators (Clements, 1983; Clements, 1984; Lean & Clements, 1981; Lowrie, 2000; Suwarsono, 1982) have shown that some students are reluctant to use visual processes when solving mathematical problems. Some of these studies (e.g. Lean & Clements, 1981; Krutetskii, 1976) have also revealed that there is a tendency for students who prefer verbal-logical (nonvisual) methods to outperform students who use visual processes. On the other hand, Lowrie & Kay (in press) found that task difficulty had a major influence on the way students represent mathematics problems.

The focus of this research was to identify how visual and nonvisual (verbal) abilities effect problem solving in mathematics. The research provides new insights into the research literature associated with problem solving because it takes into consideration students’ success in solving problems in a particular manner as well as preference for using either visual or nonvisual methods. Consequently, the placement of students along a visual-nonvisual continuum meaningfully reflects current competence and preference. The study then extends the notion of effective representation by investigating the extent to which predominant use of one method influences success in mathematics. Thus, the study examines students’ problem representation along a visual-nonvisual continuum and at either end of the continuum in relation to mathematics performance.

Method

Participants

The investigation involved 58 Year 6 students (33 males and 25 females, M = 11 years 1 month) in an Australian primary school in a large metropolitan area. The students were from two classes in the one school. Participants in the study reflected the cultural and the ethnic composition of the local community. Approximately 10 percent of the students studied spoke English as a second language.

Instruments

The following measures were administered to the students.

The *Mathematical Processing Instrument* (MPI) developed by Suwarsono (1982) was used to assess student’s preference for solving mathematical problems in either a visual or nonvisual mode. In this study, the instrument contained 20 mathematical problems and a corresponding questionnaire containing written descriptions of different solution methods commonly used when working on each mathematical problem. A mathematical problem-solving measure was generated from the number of correct responses individuals scored from the 20 questions of the MPI.

A *visuality* measure was calculated for each individual based on information obtained from the questionnaire component of the MPI. Solutions to each problem were categorized as either visual (where diagrams or drawings were used either on paper or in the mind), or nonvisual (solutions that are analytically based). For each of the 20 items on the questionnaire the following scores were derived: a) successful attempt using a visual method was allocated a score of 2; b) unsuccessful attempt using a visual method was allocated a score of 1; c) unsuccessful attempt using a nonvisual method was allocated a score of –1; and d) successful attempt using a nonvisual method was allocated a score of –2. Thus, a student who correctly answered the 20 questions in a visual manner would be allocated a visuality score of 40 while a student who correctly answered the 20 questions using nonvisual methods would have a visuality score of –40.

A *Preference efficiency* measure was also calculated from responses on the questionnaire component of the MPI. This measure was calculated in the same manner as the visuality method, except that only problems that were scored correctly counted. Thus, each correctly-solved problem was assigned a score of either 2(if it was solved by a visual method) or –2 (if it was solved by a non-visual method). Potentially, scores could range from +40 to –40 although this only occurred if a student correctly solved all items on the MPI. In order to compare students who either predominantly used visual or nonvisual methods on the MPI, the measure was restricted to students at the extreme of the visual-nonvisual continuum (ie., the 20% of students (n = 12) who had a preference efficiency measure that was in the high positive or negative ranges.

An *Item difficulty* questionnaire was used to measure students' perceptions about task difficulty of the MPI. After completing each question of the MPI students were asked to indicate, on a three-point Lickert scale, whether they felt that the question they had completed/attempted was 1) easy, 2) moderate or 3) difficult to solve. Thus, an individual's *Item difficulty* measure could range from 20 (if the student felt that all 20 questions were easy) to 60 (if the student felt that all 20 questions were difficult). The measure was completed before the students were given the questionnaire component of the MPI. Thus, the students did not know whether their solution to each problem was correct or not.

Spatial ability was measured by the Block Design subtest of the Wechsler Intelligence Scale for Children—Revised (WISC-R Wechsler, 1976). In this test participants are required to arrange a set of blocks—which are white, red or white and red—in order to match two-dimensional red and white picture designs presented on cards. Scores are given for both accuracy and speed in this particular test. It is argued that this test measures spatial visualization (Hegarty & Kozhevnikov, 1999) although such tests may also measure other spatial attributes associated with spatial relations (Clements, 1983; Lohman, 1979).

A *Numeracy* measure was calculated using the percentile ranking each student had obtained from the Educational Testing Centre (1999) *Primary Schools Mathematics Competition.* The numeracy test contained 25 items associated with number sense, 13 items associated with measurement sense and 12 items associated with spatial sense. The 50 item test was in a multiple-choice format and had to be completed in 60 minutes. The test was marked externally.

Procedure

The measures were administered in two group sessions with all students tested at once in the classroom. In the first group session, the students completed the mathematics problem-solving component of the MPI in conjunction with the Item difficulty questionnaire. After completing these two measures the children completed the questionnaire component of the MPI. The questionnaire provided descriptions of between three and five possible solution methods for each mathematical problem, and students were invited to describe their own method of solution if they had not used any of the given possible solution methods. The entire questionnaire was read to the students increasing the likelihood that the students would understand it fully. After a set of possible solutions of the questionnaire were presented orally, those students who were uncertain about the differences between two or more of the given responses for a particular question were encouraged to ask for assistance.

Less than 2 percent of responses provided by the students contained solution methods not described in the questionnaire. For these more unusual responses students were interviewed and asked to describe the method they had used to complete the problem. If the solution methods employed by the student had picture or diagrammatic (either on paper or in the mind) representations, the response was classified as visual, otherwise the response was classified as nonvisual. These interviews took place during the week of the first group session and usually took no longer than 10 minutes.

The students completed the Block Design subtest of the WISC-R on an individual basis over a two-week period. In the second session, conducted two weeks after session one, students completed the *Primary Schools Mathematics Competition* test in the prescribed time of 60 minutes.

Results

Means and standard deviations for the five measures are presented in Table 1. On average, the students solved more than 60% of the mathematics problems correctly despite the fact that the problems were originally intended for Year 7 students. It could be argued that this cohort of students was "above average" with an average percentile ranking of almost 63% for the state mathematics competition. A visuality measure of 6.2 was calculated for the group (with 42% of solutions being solved visually and 58% nonvisually) although there was considerable variation in the number of visual and nonvisual solutions employed by individuals to solve the 20 problems on the MPI. For example, one student solved all problems in a nonvisual manner while another solved all problems visually.

Table 1
Descriptive Statistics of the Variables Measured

Measure	*M*	*SD*
Mathematical processing instrument		
Mathematical problem solving	12.0	3.4
Visual preference measure	6.2	12.7
Item difficulty measure	33.5	6.3
Additional measures		
WISC-R Block Design Subtest	25.7	7.0
Numeracy measure	62.9	19.4

The first analysis considers the relationship between problem solving, problem representation, spatial ability and perceived task difficulty. The correlations between these measures are presented in Table 2. There is no significant correlation between the use of visual-nonvisual methods (the visuality preference measure) and problem-solving success.

Table 2
Correlation Matrix

	1	2	3	4	5
1. Math p.s	---				
2. Visual Mth	0.02	---			
3. Difficulty	-0.45**	-0.02	---		
4. Spatial	0.67**	-0.30*	-0.39**	---	
5. Numeracy	0.50**	0.22	-0.45**	0.53**	---

Note. Math p.s= mathematical problem solving on the Mathematical Processing Instrument (MPI); Visual Mth = visual measure on the MPI; Difficulty = difficulty rating of MPI; Spatial = Wechsler Intelligence Scale for Children—Revised, Block Design Subtest; Numeracy = NSW Mathematics Competition Percentile Ranking. *$p<0.05$. **$p<0.01$.

As Hegarty and Kozhevnikov (1999) commented, this result is consistent with other studies that have examined the relationship between representation and success in problem solving. Interestingly, the visual-nonvisual preference measure was statistically negatively correlated with the spatial ability measure. Thus, there was a relationship between spatial ability and the use of nonvisual methods on the MPI. As expected, the problem-solving measure was correlated with the numeracy measure as well as the spatial measure. Although the problems in the MPI were quite different in structure and content than the problems presented in either the mathematics-competition test or the Block test the notion of general problem-solving ability was evident. Not surprisingly, there were statistically negative correlations between the task difficulty variable and the problem solving, spatial and numeracy measures. Thus, students who considered the problems in the MPI to be easy did well in the other three tests. Similarly, students who considered these questions to be difficult did not score highly in the other three measures. There was also no statistically significant relationship between the task difficulty variable and the visuality preference measure.

The effect of gender on these five measures was then considered. Means and standard deviations by gender are presented in Table 3.

Table 3
Descriptive Statistics of the Variables Measured, by Gender

Measure	Male (*n*=33)		Female (*n*=25)	
	M	*SD*	*M*	*SD*
Mathematical processing instrument				
Mathematical problem solving	12.4	3.1	11.7	3.9
Visual preference measure	10.3	9.8	0.9	14.2
Item difficulty measure	32.6	6.9	34.7	5.1
Additional measures				
WISC-R Block Design Subtest	24.9	6.5	27.1	7.6
Numeracy measure	65.0	18.4	60.0	20.7

There were no statistically significant differences between boys and girls on problem solving [*t* (1,56) = 0.51, p>0.05, E.S = 0.2], task difficulty [*t* (1,56) = 0.13, p>0.05, E.S = 0.18], spatial ability [*t* (1,56) = 0.36, p>0.05, E.S = 0.31] or numeracy [*t* (1,56) = 0.49, p>0.05, E.S = 0.26]. There was however, a statistically significant difference between the methods used to solve the problems in the MPI [*t* (1, 56) = 2.99, p < 0.01, E.S = 0.78]. The visuality measure of M = 10.3 indicated that the boys employed more nonvisual methods than the girls M = 0.9 to solve the problems on the MPI.

Additional *t*-tests were conducted to determine whether there were differences between students with strong visual-nonvisual preference efficiency and the four measures. In this analysis the students with the highest proportion of correct nonvisual methods (20% or n=12) were compared to the students with the highest proportion of correct visual methods (n=12) from problems solved on the MPI. Thus, this variable was comparing the approach predominantly employed by students who successfully solved problems in either a visual or nonvisual manner. There were no statistically significant differences on task difficulty [*t* (1,22) = 0.53, p>0.05, E.S = 0.21], or spatial ability [*t* (1,22) = 1.0, p>0.05, E.S = 0.41] measures. There were, however, statistically significant differences between preference efficiency on the problem solving [*t* (1,22) = 2.1, p<0.05, E.S = 0.87] and numeracy [*t* (1,22) = 2.87, p<0.01, E.S = 0.73] measures. Thus, students who effectively used a high proportion of visual methods to solve problems were more likely to perform well on the mathematics problem solving and numeracy tests than students who chose to employ a high proportion of nonvisual approaches.

Discussion

It has been argued that a visual representation of a task is not usually the most efficient or effective way to solve a specific mathematics problem (Aspinwall, Shaw & Presmeg 1997; Lowrie, 1996). Nonvisual methods, based on particular algorithms or strategies, are usually more concise and efficient (Clements, 1983; Krutetskii, 1976; Lowrie & Kay, in press). In this study, strong relationships were found between problem solving and numeracy ability and yet neither measure of mathematics performance was influenced by the way in which the students represented problems.

The present study also examined the relationship between visual-spatial methods and spatial ability. Although the spatial ability measure required students to replicate patterns through spatial visualization and other spatial relations there was not a relationship between this measure and the visuality measure. In fact, the statistical negative correlation appeared somewhat unusual. This can be explained by the fact that pictorial visual imagery and other forms of visualization are quite different (Presmeg, 1992), and that the visuality measure was associated with preference for using either visual or nonvisual methods as opposed to a measure that monitored proficiency in using such methods.

There were no statistically significant differences between the performance of boys and girls on any of the mathematics measures reported in the study. In the past some studies have revealed gender differences in children’s spatial development. However this study supports recent findings (Linn & Peterson, 1985; Tracey, 1990) that indicate that such differences are becoming less apparent. In contrast, there were statistically significant differences in the way students solved problems along the visual-nonvisual continuum. Boys were more inclined to represent and solve problems nonvisually. Recent studies that have investigated the effect of pictorial imagery on problem solving have found no gender differences (Lowrie, 1996; Lowrie & Kay, in press) on task representation. In the present study both preference and success were included in the measure of a student’s visuality, which may explain inconsistencies in the research literature. Nevertheless, gender should be considered in future studies that are associated with the effect of visual-nonvisual representations on problem solving.

In this study, a more meaningful classification of students' mathematics representation was established. A preference efficiency measure was developed that considered only students who demonstrated an efficient use of either visual or nonvisual methods. There was a statistically significant difference between the performance of these students on a comprehensive numeracy test that contained questions associated with number, measurement and spatial sense. Moreover, there was a statistically significant difference between students' problem-solving performance and the preference efficiency measure. The students who predominantly used visual methods outperformed the students who predominantly employed nonvisual methods. This is in contrast to the analysis conducted with the visuality measure—where no statistically significant correlations were found between the visuality measure and the problem solving or numeracy performance measures. The visuality measure considered representational preference among students who moved between visual and nonvisual methods with fluctuations between the two methods common. Furthermore, it considered representational choice whether or not the student was able to solve the problem correctly. Most studies that have examined students' mathematical problem representation over a visual-nonvisual continuum have considered preference alone (Hegarty & Kozhevnikov, 1999; Presmeg, 1985; Suwarsono, 1982) and have not made distinctions between effective methods along the continuum.

The present study clearly shows that research concerned with the way in which individuals represent problems should examine students who use visual or nonvisual methods in an efficient manner. By investigating students who were at the two extremes of the continuum, the present study has shown that students' who were able to use visual methods regularly, and in an efficient manner, were able to solve mathematics problems across two different types of tests in a more effective manner than students' who were more inclined to use nonvisual methods on a regular basis.

The numeracy measure was taken from a relatively difficult mathematics competition paper. The format and structure of questions in this test were quite novel for these students—as they had not been accustomed to participating in external testing procedures in the past. With respect to the problem-solving measure, the complexity of the MPI (originally designed for Year 7 students) hindered students who relied too heavily on nonvisual methods. Thus, the visual methods were most effective in novel or complex problem-solving situations (Kaufmann, 1990; Pirie & Kieren, 1992). This study has shown that there is little educational value in comparing students across a visual-nonvisual continuum unless students' efficient or powerful methods are considered.

From a classroom perspective, teachers should provide students with opportunities to develop powerful visual representations when engaged in problem solving—particularly when the tasks are novel or complex. Teachers need to expose all students, including talented students who predominantly use nonvisual approaches, to a more balanced approach to problem solving.

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