Changing the Teaching of Space Mathematics

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A NSW teacher professional development project has concentrated on teaching early primary school students about Space. The project uses a research-based framework organised around two key ideas: Part-Whole Relationships, and Orientation and Motion. Within these two areas students develop and exhibit a range of strategies. Teachers are provided with background notes, task-based assessment interviews, and plans for classroom learning experiences for each area. A questionnaire, discussion groups, and lesson observations have been used to assess change and how teachers are now teaching Space lessons.

The Space Framework

The framework for *Count Me Into Space* is based on research on visualisation and spatial thinking (e.g., Eliot, 1988). Tartre (1990) emphasised that *re-seeing* was an important spatial ability when she drew together the literature on spatial abilities. It included embedding or completing shapes and disembedding or drawing out shapes from a more complex configuration. Owens (1996, 1999) extended the importance of this skill in students' learning about angles, supporting van Hiele's (1986) notion of recognition being necessary before analysis of shapes could be undertaken. Dynamic imagery is discussed by, for example., Clements and Battista (1992). Presmeg (1986) showed that high school students used pictorial or concrete imagery, pattern imagery, dynamic (or moving imagery), and kinaesthetic or body-movement imagery. Owens (1993) found that primary school students were also using this range of imagery strategies and that many were associated with specific experiences. Students in primary schools tend to develop through five groupings of spatial strategies (Owens, 1999). They provide a direction along which teachers can expect development and plan appropriate lessons.

There are two aspects of Space mathematics that we consider basic. These are (a) part-whole relationships, and (b) orientation and motion. Each of these develop with investigative strategies and imagery as well as to language and classification. It is through students' language and their selection and use of objects that decisions on students' learning is made. Table 1 provides descriptions of strategies identified for part-whole relationships. A similar table is also available for orientation and motion.

Students first begin with primitive knowing and image making (Pirie & Kieren, 1991), at which time, students are using emerging strategies. They then develop perceptual strategies which are often associated with prototypical images (see, e.g., Herskowitz, 1989). Pictorial or concrete imagery strategies are the initial mental images they can satisfactorily use. The next group of strategies involves dynamic imagery strategies and pattern imagery strategies. Students extend their conceptual knowledge and imagery by being able to use these kinds of imagery and associate them with a range of shapes, and shape and space properties. For example, a young student may see the pattern of "one up one down" for tessellating equilateral triangles or the student may mentally move one shape into another (Owens & Clements, 1998). Students also increase their visual memory, and sequential use of images, e.g, young students can mentally fold a net of an open cube (Owens, 1992). In the final grouping of efficient strategies, students are showing they know abstract relations about shapes.

Table 1
Part-Whole Relationships

	Investigating and Visualising	Describing and Classifying
	The student:	The student:
Emerging strategies	attempts to put pieces together to see what is obtained	matches shapes with everyday words e.g. ball for a circle
Perceptual strategies	recognises whole shapes used to build a shape or picture	describes similarities and differences and processes of change as they use materials
Pictorial imagery strategies	disembeds parts of shapes from the whole shape matches parts of different shapes completes a partially represented shape or simple design	discusses shapes, their parts, and actions when the shape is not present
Pattern and dynamic imagery strategies	develops and uses a pattern of shapes or relationship between parts of shapes plans and dynamically modifies a shape to illustrate similarities between different representations of the same concept	discusses patterns and movements associated with combinations of shapes and relationships between shapes
Efficient Strategies	assesses images and plan the effective use of properties of shapes and composite units to generate shapes	describes effective use of properties of shapes to generate new shapes

Teacher Development

Swafford, Jones, and Thornton (1997) attributed teacher change to increased geometrical content knowledge and research-based knowledge of student cognition. Action, reflection, autonomy and networking (Krainer, 1996) were seen as important aspects of continuing teacher development. The project provided these aspects by encouraging teachers to assess their students' cognitive development, implement appropriate lessons, then reflect with the aid of consultants, researchers and fellow teachers. Such networking was an important part of the project although there was limited autonomy since the teachers were given the tasks and lesson outlines. Teachers were encouraged to observe, listen and reflect because listening, both to students and fellow teachers, has been found to be important in teacher reflection and attitude change (Cooney

& Krainer, 1997). The assessment tasks, the lesson outlines which provided example questions and teaching points, and the teacher meetings encouraged this.

Schools and the Professional Development

Teachers were provided with a kit of assessment tasks with interview schedules, plans for learning experiences, and background notes. Three consultants provided a short initial training session on the framework and tasks, and attended three meetings during the trial and a final meeting with teachers. Notes were taken at all meetings. All 29 teachers at the five schools who were teaching kindergarten to Year 2 were involved. K and Year 1 teachers taught ten lessons on part-whole relationships and Year 2 teachers taught ten on orientation and motion but first taught five part-whole lessons. Each teacher selected eight students for assessment, four from the middle of the class, two each from the upper and lower bands of the class. In addition five non-intervention schools were selected for assessment before and after ten weeks and students' results were compared. Students improved significantly more in the intervention schools (reported in Owens, Reddacliff, McPhail, submitted).

The teachers completed, before and after, a 26 item questionnaire on teaching space mathematics that covered knowledge, approaches, and attitudes to Space mathematics. Six teachers, one at least from each school, volunteered for their lessons to be observed by a researcher. These teachers generally felt confident about their teaching and were very experienced. Three teachers had been involved with consultants' visits and short-training for the NSW DET early number program. Except the younger teacher, none had participated in Space mathematics in any inservice or degree in the past 10 years.

Detailed notes were taken during the lessons and used to address specific questions such as what language did the students use, how did students manipulate materials, what were the students encouraged to do, what difficulties did the students have, what growth in children's thinking was observed, and what questions did the teacher use. Aspects of lessons that varied between teachers were analysed and provide several themes for discussion.

Results

Focussed Discussions and Meetings

From the first interview, it was clear that teachers previously concentrated on naming standard shapes such as triangles and circles and ensuring students could repeat basic properties such as "it has three sides". The ideas of part-whole relationships and orientation and motion made sense to the teachers although these ideas were not a focus of their teaching at the beginning of the project.

Eighty-six percent (86%) of teachers involved in the project had participated in professional development in mathematics education during the last two years, mainly through participating in the Count Me In Too project. However, 89% of teachers had not experienced professional development in space mathematics during the last ten years. Nevertheless, teachers said that the intervention program extended what they already did with the Syllabus activities. Only gradually did teachers show an increased understanding of part-whole relationships and orientation and motion. They began to talk more of the different kinds of strategies that they saw students using during the intervention and related easily to the descriptors during assessment except that they were concerned if students did not improve on all tasks despite the differences in the tasks.

Teachers found that their young students were able to do more than previously expected. Students were able to learn about properties of shapes, angles and new terminology. Students were able to move away from stereotyped concepts of shapes and investigate how shapes can move and change. The lessons allowed students to take risks and manipulate a variety of objects. A typical comment was, “Many of the concepts and terms I thought we were introducing too soon and they wouldn’t be ready for them, but they loved it.”

Teachers felt that the lessons allowed them to focus more on what students were actually doing. The lessons allowed students to take risks and manipulate a variety of objects. Significant improvements in students’ use of the language of space mathematics were noted. One teacher commented, “They’re a sponge waiting to happen. They loved it. When I used the term “flipping”, they took off. They loved flipping and sliding.” Another teacher said, “I found that students were reflecting more and listening more to each other and this developed their understandings further.”

The lessons were enjoyable, useful for extending students and teachers and were appropriate for a range of abilities. One teacher said that she couldn’t wait to see what the next lesson was. Teachers were given freedom to choose the lessons with which they felt comfortable and had appropriate resources in the school. Having engaged students in the lesson, they felt free to allow students to develop the direction of the lesson. One teacher said that the lessons were an entry-point for a more open-ended style of teaching.

Questionnaire Responses

Pre- and post-intervention questionnaires showed that teachers changed several attitudes and practices during the intervention period. Following the intervention, teachers were more likely to devote three out of every ten mathematics lessons to teaching space (increased from 59% to 80%), and made more equipment (increase from 21% to 40%). A greater percentage of teachers felt that they knew more about how children learn about space (increased from 7% to 40%), and they were more satisfied with their teaching in the space strand of mathematics (increased from 28% to 72%). However, it seems that some items that might have been expected to change positively did not and that this was somewhat inconsistent with verbal feedback. Fewer teachers agreed that visual imagery included moving and patterned images. If these results are correct (decrease from 90% to 64%), then teachers may have read the question in more than one way (e.g., that her class of children were less likely to be using moving and pattern imagery), or otherwise they were not understanding the framework or had little idea of what the question meant at first.

Classroom Observations

The following features of lessons were found to be important themes in lessons:

- The selection of activities
- Students remaining engaged
- The exploratory, creative, and open approach to the lesson
- The questions that were used
- The expected language
- Students listening to each other in the feedback session.

Each of these is illustrated below.

The selection of activities. Some lessons taught at the beginning of the project can be compared to later lessons. For example, in a Year 1/2 lesson, the teacher discussed symmetry of the square and cutting it up to make equal halves. She then gave each student a square of paper with a picture of the side of a snail, without symmetry, and students were asked to cut it up and join it together. The final discussion asked questions such as which cut did you do first? So what shapes did you have? How many pieces did you end up making? Who cut diagonally first? These questions needed only one word answers and did not draw out the purpose of the lesson. Students tended not to listen to other students' responses during the final sharing time of the lesson.

During a Year 2 lesson, students worked at their desks individually to make squares on geoboards. Each step of the lesson was carefully directed by the teacher. After being directed to make a square, the teacher suggested other orientations, overlapping squares, and discussion of properties of squares. The lesson extended some students who had trouble making squares in different orientations and transferring their squares onto dot paper or who showed growth on the properties of squares. Nevertheless, other students were not extended. The teacher asked some reasonable questions. "Did anyone find any other shapes when you overlapped some of the squares?" To which children answer "triangle", "quadrilateral", and "trapezium". While the teacher does not extend the conversation by asking for comparisons to the square, she did ask how students knew they had made squares and how they might transfer them to dot paper.

The students remaining engaged. In the last lesson above, students were given time to make many different types of squares, but many became bored and did not listen to each other in the final feedback session. The same learning experience was given to a composite K/1 class with different results. It was more challenging to them.

In one class, students were given the opportunity to make their own shapes on a folded line, they remained engaged. Different shapes were encouraged. Students were encouraged to talk about what they were doing. They expected to be able to talk about their shapes at the end of the lesson. In another lesson, students were encouraged to make their own triangles on a given triangle. The challenge of making overlapping triangles kept students attention although some lost interest when they had to colour in as this took a long time. Students' talk was more about the mathematical concepts when they were engaged.

The exploratory, creative, and open approach to the lesson. In a K/1 class, the students began with an open-ended activity making and describing different shapes on the geoboards and then they looked at properties of the square. While this is similar to the earlier lesson, the students worked in pairs and were more inventive initially, described their shapes with a variety of words, and then they concentrated on what makes a square special and the students are directed to make squares.

T: Close your eyes and imagine a square in your mind. Who can tell me about a square?
S: It has four sides.
S: It has one face.
S: It has four points.

After making squares, they discuss the various squares and those that are not squares, usually by counting pegs. Different squares are expected but the common feature of equal sides was noted. When a student responds that one shape is a rectangle and another student that it is a quadrilateral, the teacher says "Yes, it is a rectangle and a quadrilateral because it has four sides. Who can help Lee to make a square?" Students then explored the properties by making several different squares.

The questions that were used. A closer look at the questions and how they fitted in with the approach shows how this last lesson differed. There is a general atmosphere of being creative and to explore while still developing ways of talking about the shapes and using analysis to check. It was appropriate for the class as it was a challenge.

In a Year 2 lesson on symmetry, a teacher encouraged prediction, variation in recording, and the students to be adventurous in designing shapes. Students showed that they understood shapes to be dynamic and enjoyed making different shapes by predicting what they would get by cutting on a fold line. Instructions were clearly presented at the start of the lesson. Several students were given the opportunity to answer a question and the students listened to each other and added new points. Questions made students think:

T: How can we fold it in half?
S: Diagonally.
T: How do we do it diagonally?
S: Like this (student indicates diagonal line from corner to corner using finger).
T: How do you know that it's folded in half?
S: Because it's turned into a triangle.
S: Because they're both exactly the same.
T: Is there any other way we can fold it?
S: Down (student indicates a vertical line from the middle of the top side to the middle of the bottom side) ...
T: How else can we fold it in half?...
T: Now I'm going to draw a shape on the folded square. (uses texta to draw a half triangle shape and the paper is clipped to the white board) Can anyone predict what this shape will look like when it's cut with scissors and opened up?
S: Triangle...(the students explain why and draws the predicted finished shape)...
T: This time I'm going to fold it in half and I'm going to draw a different shape on it. Remember the shape has to start and finish on the fold. What shape will I draw?
S: Not a usual shape.
T: There's my shape. Who can predict what it will look like if I cut it with scissors around here and open it out?
S: It will look like the end of a ribbon.
T: Yes, it does look a bit like the end of a ribbon. Can you tell me about the sides?
S: They go like that (student indicates the outline in the air using index finger).
T: How many sides are there?
S: Five.
T: How do you know there's five?
S: Because you can't count that (indicates the fold) because when you cut it there, it won't be there.
T: How many sides will there be?
S: Eight.
T: How do you know?
S: Because it's half (four sides are visible).

After this long discussion at the start of the lesson, the students are given scissors and squares and discuss with their neighbours what they are going to do and then they try it out. During this time the teacher asks questions such as the following.

T: When you fold it this way, what does it mean?
S: Symmetrical.
S: Horizontal.

At the end of the lesson, the students share again.

T: How did you fold it?
S: Vertically. Then I looked at the half on it. Then I traced it on the side. Then I imagined it. Then I cut it out.

> S: I folded it horizontally. Then I thinked [sic] about it. Then I drew it. Then it didn't turn out symmetrical so I had to change it.

Further questions and answers followed.

By contrast, sometimes the questions were very much on factual information, for example: What shape is it? And students' talk tended to be names. Final questions were sometimes not close to the purpose of the lesson as illustrated by the first lesson described above.

The expected language. In a kindergarten class, the students were set fairly open investigations. After a lengthy introduction with the teacher making shapes with her body or students demonstrating, students made shapes with their bodies, and then they use a length of string to make shapes. The students used many shape names and discussed how they changed shapes e.g. rotated it, stretched it, the corners need to join up. In this lesson, the type of activities and the group work often drew out the language more than the teacher's questioning.

In the kindergarten lesson in which students were drawing triangles on a triangle, the teacher encouraged students to use parts of the given triangle to draw triangles. Different shapes were made and discussed. In a previous lessons students had drawn individual triangles, without overlap on the given triangle. It was clear that the teacher's initial attempt to teach from the provided plans had not satisfied her. She was asking questions like "how do you know it is a triangle? And think of some ways to make different triangles. And why isn't a hexagon the same as a triangle? The students used names of shapes and parts but also showed knowledge such as triangles have three straight sides.

Students listening to each other during the feedback session. In the lesson in which students made their own symmetrical shapes, the students listened to each other and extended one another's ideas.

Conclusion

The learning experiences provided to teachers and their regular reflective meetings did encourage teachers to recognise what they were doing well in their teaching and how they might improve their teaching. It was, however, more difficult for some teachers to grasp the key ideas of the framework and develop creative activities and questions that encouraged the students' use of imagery.

The lesson observations suggest that some changes are needed in the management of activities. Teachers need to be able to structure the activity to allow students to explore ideas. This requires preliminary whole class discussions which encourage predictions of what might happen by a given action, and a variety of responses. The key concept such as symmetry needs to be clearly established in the discussion and the activity. Students also need to value one another's comments. Teachers should not be seen as the only fount of knowledge in the classroom which was a tendency in over-directed lessons. Questions need to challenge beyond the one-word response and to encourage students to share their thinking. Students need to remain interested in the task until the end of the session. This sharing session should be structured so that students can build on one another's comments.

Teachers may also need to reconsider how they use stencils. A stencil that they used before may not fit into the purpose of a lesson. It might cloud the main idea or discourage open, creative activity. When some students are capable of exploring further ideas about a shape, a simple task may need to have the key idea such as all the important features of all squares explored beyond the basic fact of each individual square having equal sides. This can be achieved by comparison and by drawing out similarities and differences between the shapes that the students have created. One particular strength of all the lessons was the hands-on making and using of shapes which students enjoyed discussing.

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