From Geometric Patterns to Symbolic Algebra is Too Hard For Many

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There are well-known problems with using patterns as a means of introducing algebra. This paper presents empirical data to support the view that it is inadvisable to try to introduce students to algebraic conventions by moving directly from patterns into algebra notation. These data also support the logical claim that students need familiarisation with the language of basic algebra prior to being expected to express pattern generalisations in symbolic form.

This paper is presented in order to:

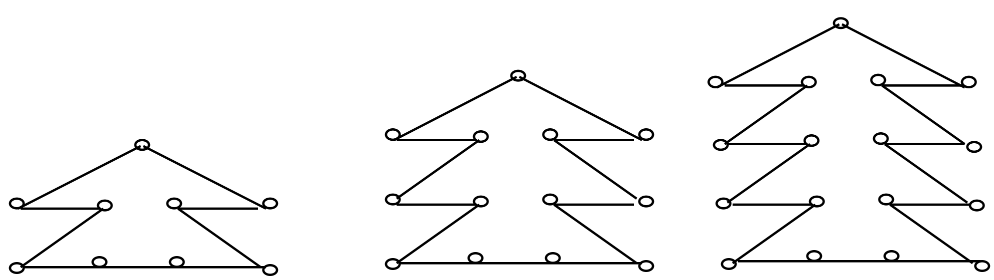
1. share with the mathematics education community data from several sources to underscore the reality that many students have great difficulty using algebraic symbolism to express generalisations derived from extendable geometric patterns;
2. recall some of the theoretical explanations for such difficulties;
3. consider current educational practice in relation to the nominated problem; and
4. recommend that educators, syllabus writers, and publishers respond appropriately.

Empirical Evidence That Many Students Have Difficulty Expressing Generalisations in Symbolic Algebra

Data from Test Item A. Test Item A was used for measuring students' ability to write algebraic equations in a research project by MacGregor early in 1996 and, later that same year, in another project by Quinlan. The item was as shown in Figure 1.

The picture shows the lights on Christmas trees of different sizes. Complete the table.

Tree size	2	3	4	5	6	...	...	...	10
Number of lights	9	13	17	...	...	...	...	...	...



SIZE 2 SIZE 3 SIZE 4

(a) How many lights would you need for a Size 16 tree?
(b) How many lights would you need for a Size 100 tree?
(c) Explain in words how you can work out the number of lights when you know the tree size.
(d) Use algebra to write an equation that relates the tree size and the number of lights. [Use s for the tree size and n for the number of lights.]

Figure 1. Test Item A.

Table 1
Number Correct on Writing Equations and Stating Solutions (Item A, Quinlan Study 1996)

Item A	Comment	Totals (*N* = 67)
(b)	arithmetic	28 (42%)
(c)	verbal	27 (40%)
(d)	symbolic	21 (31%)
(c)+(d)	both correct	19 (28%)

Overall, as Table 1 records, only two-fifths of the students in the Year 7 classes in Quinlan's 1996 study gave a correct verbal generalisation in Part (c), and less than one-third of them succeeded in writing a correct equation for Part (d). Still fewer had both of these parts correct.

Table 2
Comparison of Percentages Correct on Part (d) for Equation-Writing

Quinlan, 1996 (Year 7)	MacGregor, 1996 (Year 8)	MacGregor, 1996 (Year 10, School A)	MacGregor, 1996 (Year 10, School B)
31	30	35	72

Table 2 shows the percentage success rates of students in both 1996 studies. The point of reporting these figures is that (apart from one group of Year 10 students) only around one-third of students had success with writing the required equation in algebraic symbols. For many students, then, in these two studies, moving from a geometric pattern to symbolic algebra proved too hard.

Data from Test Item B. In 1998, Quinlan was one of a team researching learning/teaching strategies for early secondary school (Quinlan, 1999). A large number of students contributed data as teachers from 23 schools responded to invitations to participate in the study. The sample included 50 teachers and their 1156 Year 7 students across 53 classes from Government schools, private schools, church schools - central schools, large schools, city and country - from NSW School Districts and Sydney metropolitan - plus two schools in Victoria, one Government and one private. The students were in their first year of Secondary school and the average age was 12.64 years (ranging from 11.33 to 15.17 years). All followed a mathematics program which covered the strands Problem Solving, Geometry, Measurement, Statistics, Number, and Algebra. Students twice completed a test covering many topics for Year 7, first at mid-year ("pretest") and then towards the end of the year ("posttest"). Between the two tests students followed their normal program of mathematics lessons with their usual classroom teachers. Responses to Item B from the test (Figure 2) identified student difficulties.

The most obvious feature of these outcomes is the dramatically low success rate for moving from a geometric pattern to a verbal generalisation and then to the use of algebraic symbols to express the generalisation. The persistence of the difficulties is shown by the fact that success in the pretest did not ensure success in the posttest.

The following pattern of linked upright triangles is built from match sticks:

1 upright triangle, 2 linked upright triangles, 3 linked upright triangles, ...

(a) In this pattern, how many sticks for 1 upright triangle? _____ 2 linked upright triangles? _____
3 linked upright triangles? _____ 10 linked upright triangles? _____

(b) Write a sentence to describe to a friend how to work out the number of sticks needed to build whatever number of linked upright triangles he or she might think of. ________________.

(c) How many sticks would be needed to build *K* linked upright triangles? ________________.

Figure 2. Test Item B.

Table 3
Statistics for Correct Responses to Test Item B

Criterion	Pretest (N = 1009)		Posttest (N ≈ 760)		Both tests (N ≈ 640)	
	n	%	*n*	%	*n*	%
(a)(iv) correct	355	35.2	298	38.8	130	20.2
(a) all correct (arith.)	343	34.0	287	37.5	125	19.4
(b) correct (verbal)	117	11.6	109	14.3	24	3.8
(c) correct (symbolic)	40	4.0	65	8.6	9	1.4
(b) & (c) both correct	30	3.0	45	6.0	2	0.3

Data from Test Item C. Across the State of New South Wales, Year 10 students, in their fourth year of Secondary School, complete a common mathematics examination for their School Certificate. In 1998, a question (Test Item C) towards the end of the paper for more than 78,000 candidates centred on a pattern embedded in the number of fence posts needed for rectangular strips of land. The posts were 2 metres apart and the dimensions of the rectangles increased by 2 metres at a time, starting with a shape 2 metres by 4 metres. The success rate varied as follows for the different parts of the question.

In Part (a) 72% successfully extended the pattern by drawing the next diagram. Part (b) asked them to “Complete this sentence to describe the pattern in words: Beginning with the number 6, …” and the success rate was 49%. When asked for the number of posts needed for a field “whose longer side measures 14 metres”, 30% succeeded. Part (d) was found to be the hardest. It asked for the number of post needed “for a strip of land whose longer side measures k metres (where k is an even number)”. Only 15% were correct, and only 17% were able to calculate the area of the rectangle that needed 58 posts. The markers’ comments on Part (d), which required a generalisation in algebraic symbols were:

- This part was quite difficult for many candidates.
- Many candidates gave a numerical answer.

- Some candidates wrote their answers around the wrong way, i.e., $p = k/2 + 1$ or else gave their answer as a highly complicated algebraic expression. (Board of Studies, 1998, p.25)

Data from Test Item D. The last question in the 1999 School Certificate Mathematics Examination centred on the geometric pattern formed by a line of cubes, each of which had two dots on each face. Around 81,000 candidates were asked the state the number of dots on the outer surface when the number of cubes joined in the line was (a) 4; (b) 7; (c) *n*. The percentage success rates were as follows: (a) 66% correct; (b) 53% (c) 18%. The examiners' comments about Part (c) read:

- Most candidates found this part very difficult, and did not seem to understand what was being asked of them. The concept of an algebraic expression seemed to have eluded these candidates, with comments like 'what is n?' written in the answer space. (Board of Studies, 1999, p.44)

Thus, even for students aged around 15 or 16, it is a difficult task to use algebra to express a generalisation about a geometric pattern.

Theoretical Explanations for Student Difficulties

Verbal Generalisation Before Symbolic Generalisation

In the Quinlan 1996 study, of the 21 students who wrote the equation $n = 4s + 1$, 19 had correctly described the relationship as "You multiply the size by 4 and add 1" (or equivalent). However, another eight students correctly described the relationship in words but were unable, at this early stage in their experience with algebra, to translate this into a symbolic equation. Thus, Item A provided support to the conclusion made by MacGregor and Stacey (1993) that "students who gave a correct verbal description were more likely than other students to write a correct algebraic rule" (p. I-186). They added that "Our findings suggest that the verbal description is an important and perhaps necessary part of the process of recognising a function and expressing it algebraically" (p. I-187).

Difficulty Linking Visual And Symbolic

These outcomes highlight two possible explanations for student difficulties, namely the difficulty of being able to express verbally a generalisation derived from a pattern, and that of moving on to a symbolic form of the generalisation. Warren (1992) points out the first of these: "endeavouring to link visual and symbolic representations is important and yet it seems that the majority of students are failing to achieve this goal" (p. 253). She also found, from semi-structured interviews of 36 students, that students "find it easier to describe generalisations in tables of data than in visual patterns" (Warren, 1997, p. 566).

Language Skills

MacGregor and Price (1999) have alerted researchers and educators to a second line of thought, namely, that "it is possible that students' poor understanding of symbols and syntax in algebra reflects inadequate symbol and syntax awareness in ordinary language" (p. 453). Their analyses of data collected from 1236 students aged 11 to 15 years across 18 schools in Melbourne showed that "in each grade, all students who obtained maximum or near maximum scores on algebra also obtained maximum or near maximum scores on language items" (p. 456). In their follow-up study (Study II) using more difficult language items with 340 Grades 8 to 10 students, whose first language was English, they found that "high language scores tended to be associated with high algebra scores" (p. 459). They were aware that bilingual students might well have a "cognitive advantage" (Lambert, 1978, p. 217). Indeed, a study in Hong Kong revealed that the syntactic structure of Chinese seemed to play an important role in accounting for better performance by first-language Chinese students than first-language English students on algebra questions such as "p and q are numbers. p is 6 more than q. Write down an equation that describes the relation between p and q" (Lopez-Real, 1997, 318). There is still much to learn about the links between language competence and competence with algebra. As MacGregor and Price (1999) commented, "We do not know why, in both Study I and Study II, some students with good language scores had many algebra items incorrect although they had been given the same opportunity to learn as their classmates" (p. 461).

Language of Algebra Needed First

The 1998 Quinlan study students were asked: What is the value of $2q - 5$ if $q = 9$? Their success rates for showing an understanding of the algebraic expression $2q - 5$ were 25.2% in the pretest and 48.8% in the posttest. In the pretest more than 45% of the Year 7 students gave the incorrect answers 24, 4, or 6, showing that they misunderstood the conventions built into the expression $2q - 5$. More than 34% of them did likewise at the end of the year. The message is that understanding first degree algebraic expressions is not all that easy for beginners.

This shows that, despite the statistically significant improvement during the second half of the year, many of these algebra beginners were struggling with what may be called *the language of algebra.* It would be clearly unreasonable to ask anyone to translate the statement "The number of sticks equals one less than four times the number of upright triangles" into Swahili or Cantonese before they were fluent with the nominated language. Similarly, translation of such a statement into algebraic symbols needs to be preceded by fluency with the language of algebra.

Inability to substitute correctly in these first-degree expressions reveals a lack of assimilation, in the Piagetian sense, of the conventions for using the language of algebra for first degree expressions. It is logically unwise to expect students to use algebraic symbolism for their pattern generalisations (for Item B it was $4K – 1$, or an equivalent) if, as many were found to be, they were unable to interpret correctly an expression such as $2q - 5$.

A powerful aspect of the Quinlan 1998 data in helping to unravel causes of student difficulties with expressing generalisations in symbolic algebra was the following. Those who succeeded with the symbolic form of the pattern generalisation were usually those who could at least understand the meaning of $2q – 5$. Of the mere 36 students (out of 782) who had Part (c) of Item B correct in the pretest, 27 (75% of them) had correctly interpreted $2q - 5$. Furthermore, of the 64 correct (out of 658) in the posttest, 51 (80% of them) had correctly interpreted $2q - 5$. These outcomes support the recommendation that, before asking students to write their generalisations in algebraic symbols, students need to become familiar with the language of algebra.

Current Educational Practice

For many years, a popular ingredient for programs designed to introduce students to algebra is the formulation of generalisations based on patterns. The important step of relating two variables embedded in the pattern is usually expressed in ordinary language, resulting in an algebraic statement without the use of algebraic symbols. This is to be applauded, as the step of expressing such a relationship is cognitively demanding, even without recourse to algebraic symbols. To illustrate the popularity of this process, examples are now quoted from influential sources across three continents, and examples are given of recent textbook approaches.

Firstly, The South Nottinghamshire Project in the U.K. during the mid-70's resulted in the production of Journey into Maths. In the chapter of the Teacher's Guide 1 devoted to Sequences and Functions, we find:

> Starting from some simple geometric situations set up with matchsticks, pegboards and other simple apparatus, we can generate tables of values … Attention is drawn to the functional aspect of the table by making a jump to large values. A rule which relates each number to the number below it provides a direct way of completing entries in the table. Functional relationships are stated firstly in verbal form, with short-hand notations also being considered. (Bell, Rooke, & Wigley, 1978, p. 58).

The National Council of Teachers of Mathematics in U.S.A. had this to say in their 1989 publication for Curriculum Standards for Grades 5 – 8:

> Activities in grades 5 – 8 should build on students' K – 4 experiences with patterns. They should continue to emphasize concrete situations that allow students to investigate patterns in number sequences, make predictions, and formulate verbal rules to describe patterns. Learning to recognize patterns and regularities in mathematics and make generalizations about them requires practice and experience. Expanding the amount of time that students have to make this transition to more abstract ways of thinking increases their chances of success. (National Council of Teachers of Mathematics, 1989, p. 102).

The same principles are re-iterated in their Standards 2000 publication (National Council of Teachers of Mathematics, 2000) in the first entry for Algebra Standard for Grades 6 – 8:

> In grades 6 – 8 all students should represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules.

In Australia in the early 90's, the Curriculum Corporation produced the series *Access to algebra* (Lowe, Johnston, Kissane, & Willis, 1993), a series to which the present writer contributed. In the first unit in Book 1, Pattern, students are asked to write sentences to describe patterns and state functional relationships in exercises such as:

> Discuss the pattern with your partner and write down a description of it in words (p.5)

> What is the rule which connects the length of the side to the perimeter? (p.8)

> Write a brief report explaining your rule for finding the number of matches needed for each shape number and why it always works. (p.22)

Different ways of building given geometric patterns are shown to result in different formulations for equivalent rules.

Finally, a specific example from one Australian State. The present writer worked for some four years with a team to revise the Years 7 and 8 Mathematics Syllabus for New South Wales leading to its implementation across the State in 1988. The introduction to the Algebra Section, A1, was entitled *Algebra without symbols.* This encouraged teachers to have their students "express, in different ways, written sentences which describe geometric and number patterns", and "to describe, in their own words, relationships contained in those patterns" (p. 107). The next section was called *The language of algebra* and, while it pointed out that "Early treatment of algebraic symbols could flow from language used to describe counting patterns experienced in A1" (p.108), it included the following warning and suggestion:

> Using generalisations about geometric patterns (as in A1) may give rise to algebraic expressions that are very complicated. Hence it may be helpful for students to experience a range of concrete representations for algebraic expressions such as those which follow. (p.108)

Three types of concrete representations were suggested, namely,

- an area model in which the chosen value for a variable could be represented by the number of square centimetres of area in a chosen shape;
- a length model in which variable numbers of centimetres of length could be used; and
- the objects-and-containers model.

For each of these concrete models, teachers were reminded that "It is important to develop an understanding of the use of letters as algebraic symbols for *variable numbers* of objects rather than for the objects themselves" (p. 109).

Recent Textbook Approaches

A study of 11 mathematics textbooks published since the start of 1987 for Year 7 students in New South Wales, revealed that all made use of patterns when introducing students to algebra. Only two of them spent some time on the use of alphabetic symbols for variable numbers before leading students to express generalisations from patterns in symbolic algebra. The other nine textbooks went directly from descriptions in words of generalisations derived from patterns to the rewriting of these in algebraic form. Subsequently, more work was done with algebraic expressions. Five of the books used the objects-and-containers model to help develop an understanding of the conventions for first degree algebraic expressions, and one used an area model for the same purpose.

Practical Implications and Recommendations

The outcomes identified in this present study underline the fact that most students in their first year of Secondary School, and even later, find it very difficult to express in algebraic form a generalisation derived from a geometric pattern. Well-structured, sound sequencing is needed to improve the success rate. In particular, efforts need to be included to ensure that students are comfortable with algebraic notation before being expected to write algebraic generalisations in symbolic form.

It is recommended that considerable time and effort be devoted to helping students understand and become comfortable with the conventions of algebraic notation before expecting them to convert their verbal generalisations into algebraic symbols. The verbal generalisations could well be integrated with the introductory phase of leading students to take the cognitive step of speaking and writing about number in general. Patterns can help them to think about, talk about, and write about numbers in general. Hence, the use of patterns is worthwhile if it helps students to develop their understanding of the abstraction we know as a *variable*. The challenge and importance of such a concept was spelt out by this author's doctoral research project: "The major challenge for secondary students appeared to lie in developing the concept of a numerical variable" (Quinlan, 1992, p. 342).

One follow-on from this research project could be that publishers, producers of CDs for mathematics education, and teams working on syllabus revisions might heed the advice given in the 1988 New South Wales syllabus about the need to spend time on the meaning of algebraic conventions and symbolism *before* moving students to translate their generalisations from ordinary sentences into symbolic algebraic statements. This is assuming, of course, that the use of patterns is maintained.

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