The Arithmetical Strategies of Four 3rd Graders

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Four 3rd-grade students were individually interviewed and each interview was videotaped for subsequent analysis. The interviews involved presenting addition and subtraction tasks involving 1- and 2-digit numbers, and early multiplication and division tasks. Detailed descriptions of students' strategies are provided. Three of the students all used just one kind of strategy, that is counting by ones and using fingers to keep track. The fourth student used a range of strategies including adding through ten and using a known fact.

Number in the early years of school (4- to 9-year-olds) has been an area of prolific research in the last 25 years (e.g. Fuson, 1992; McIntosh & Dole, 2000b; Mulligan & Mitchelmore, 1996). Research has focused on a range of aspects of the learning and teaching of early number including children's addition and subtraction strategies (Beishuizen, 1993; Beishuizen & Anghileri, 1998; Boulton-Lewis, 1998; Carpenter et al., 1997; Cooper, Heirdsfield & Irons, 1995; Gray, 1991; Gray & Tall, 1994; Heirdsfield, 1998; 1999; McIntosh & Dole, 2000a; Thompson, 1994, Wright, 1994) and the development of early multiplication and division (Mulligan & Mitchelmore, 1997; Steffe, 1994) and early fraction knowledge (Hunting & Davis, 1991).

In the last six years, school systems in New Zealand and Australian states and territories, have funded major research-based initiatives focusing on mathematics in the early years of school (e.g. Australian Numeracy Research Projects, 2000). These initiatives have drawn significantly on the research referred to above. Several of these initiatives involve teachers using learning frameworks (e.g. Jones et al., 1996; Mulligan, 1998; Wright, 1998) in assessing children's knowledge and strategies as a key initial phase of associated teacher professional development programs. Typically this involves teachers undertaking individualised assessments which in some cases are videotaped for later analysis (NSW Department of Education and Training, 2001; Wright, Martland & Stafford, 2000).

Focus and Methodology

The research reported in this paper constitutes an aspect of the on-going research and development associated with the NSW Count Me In Too (CMIT) systemic initiative (e.g. Bobis & Gould, 1998; 1999; 2000). A basic goal of CMIT is to enhance teachers' understandings of young children's arithmetical strategies. Since its commencement in 1996, CMIT has involved approximately 1000 schools, that is approximately 55% of schools in the system. This research had the practical goal of trialing and refining a new interview protocol referred to as SENA2 (Schedule for Early Number Assessment). The focus of SENA2 includes more advanced addition and subtraction strategies, early multiplication and division, and knowledge of tens and ones, and the focus of SENA1 includes students' knowledge of number words, numerals and early counting, addition and subtraction. SENA2 is typically used in grades 2-4, and SENA1 is typically used in grades K-2. The research had the theoretical goal of providing detailed descriptions of the arithmetical strategies of 3rd graders (8- and 9-year-olds) and investigating possible links between children's early multiplication and division strategies on one hand, and their strategies for addition and subtraction involving 1- and 2-digit numbers. Specifically the research had the following goals:

1. to develop detailed descriptions of the strategies used by several 3rd graders to solve addition and subtraction tasks involving 1- and 2-digit numbers, and early multiplication and division tasks; and
2. to investigate for each student, possible links between early multiplication and division strategies and strategies for solving addition and subtraction tasks.

This report focuses on four of a total of eight students who were interviewed (duration approx. 25 minutes). The interviews were videotaped for subsequent analysis. This analysis involved writing detailed descriptions of students' strategies via an approach developed and used extensively by Steffe and others (e.g. Steffe & Cobb, 1988; Cobb & Whitenack, 1996). The interviews and analyses were conducted by the author.

Results and Discussion

The four students were presented with several addition and subtraction tasks involving 1- or 2-digit numbers. Most of these tasks were presented in horizontal, written format. The students were also presented with multiplication and division tasks in various forms including arrays and horizontal written format. The following section presents descriptions and discussions of the strategies used by each student to solve some of the tasks.

Sally

Addition with a 1-digit addend. Sally was asked to work out 19+3 which was presented on a card. She looked ahead for three seconds with her hands under the desk and then answered "22". When asked what she had done she said, "Counted on my fingers". A likely explanation is that Sally counted subvocally "20, 21, 22", and used her fingers to keep track of her three counts. Presumably Sally stopped counting at "22" because she had constructed a finger pattern which for her, symbolized "3". This strategy can be described as "counting-up-from using fingers to keep track".

Subtraction in the range 1-10. Sally was then asked to work out 7-5 presented on a card. After one second Sally answered "2". Again her hands were under the desk. When asked to explain her solution Sally said, "I got 7 fingers and took away five". At this point in the interview Sally was asked to put her hands on the desk. She solved 9-2 and 9-7 by first raising nine fingers, lowering the number equal to the subtrahend, and then counting those remaining. In summary, Sally's strategy for subtraction in the range one to 10 was to establish a finger pattern for the minuend, lower the number corresponding to the subtrahend and then to count those remaining.

Subtraction with minuend in the range 11-20. Sally was then presented with the following three subtractions; 11-2, 14-2, and 14-8. Her strategy was analogous to her "counting-up-from" strategy described above and can be described as "counting-down-from using fingers to keep track". On the task 14-8 for example, she counted from "13" to "6" in coordination with raising 8 fingers. Presumably she stopped counting at "6" because her finger pattern symbolized "8".

Addition involving two 2-digit addends. Sally used "counting-up-from" to solve 43+21 and 37+19. On the first task she raised each of her ten fingers in turn, while counting from 44 to 53, and again while counting from 54 to 63. Finally, she raised one finger in coordination with saying “64”. Presumably her last raised finger symbolized 21 counts (or the 21st count). The second task was solved in similar vein.

Quotitive division involving a partially screened array (12÷2). This task involved a 6X2 array, with five rows screened. The interviewer told Sally that there were 12 dots in all and asked her to figure out how many rows of two there were. Sally’s solution involved counting subvocally from “1” to “12”. For each pair of number words she coordinated the first and second words with raising her left point finger and thumb respectively. As well, she coordinated each pair of number words with raising a finger on her right hand. After her last two counts — “11, 12”, she lowered the 5 fingers on her right hand and then raised her thumb, looked up and answered “6”.

Multiplication in written format. Sally solved “8X4” presented in horizontal written format. Apparent in Sally’s solution was that she counted subvocally to 32 and used a finger pattern for 8 three times to keep track of her counting. Equivocal to some extent is whether she started her count from “1” or “9”. Starting from “1” is the more likely explanation because in her explanation she said, “I counted to 8 and then I counted 3 more”.

Tania

Addition involving a 1-digit addend. Like Sally, Tania used counting-up-from using fingers to keep track, to solve 19+3.

Subtraction involving a 1-digit minuend. In an ensuing discussion of her solution to 7-5 Tania indicated that she could solve this task either by making a finger pattern for 7, lowering 5 fingers, and counting the remaining two; or by counting-down-from using fingers to keep track. Of these two strategies the first is the same as that used by Sally to solve tasks of this kind, and the second is the same as that used by Sally to solve subtraction tasks with the minuend in the range 11-20.

Addition involving two 2-digit addends. In attempting to solve 43+21 Tania counted subvocally with her arms folded for 8 seconds and then answered “54”. In an ensuing discussion of her method Tania demonstrated her strategy as follows. She counted from “44” to “48” in coordination with sequentially raising the fingers of her right hand. She then counted from “49” to “53” and again used the fingers of her right hand. Finally, she raised her right thumb saying “54”. On the task of 37+19 Tania counted subvocally with her arms folded for 9 seconds and then answered “51”. When Sally attempted to solve 43+21 she made 11 counts rather than 21 and when she attempted to solve 37+19 she made 14 counts rather than 19. A viable explanation of her solution attempts is that on both tasks she incorrectly regarded 5 counts as 10 counts. Thus on the first task she regarded two fives and a one as two tens and a one, and on the second task she regarded two fives and one four as one ten, one five and one four. Consistent with the hypothesis that Tania regarded five counts as ten counts is that she apparently used her right hand only when keeping track of counting by ones. Her strategy can be described as “counting-up-from using the fingers of one hand to keep track”.

Subtraction involving two 2-digit numbers. In attempting to solve 67-38 Tania counted subvocally for 28 seconds and then answered “54”. A viable explanation is that she correctly kept track of 8 counts but did not correctly keep track of 30 counts. As in the case of addition involving two 2-digit addends, she regarded five counts as ten counts, and she did not correctly keep track of the number of tens she had counted. She might have made

13 counts rather than 38, and regarded two fives and three as constituting 38. Although given the relatively long time that she counted subvocally, it seems more likely that she made 23 counts — regarding four fives and three as three tens, one five and three, and also made a number word sequence error. In an ensuing discussion Tania indicated that she used the fingers of her left hand to keep track of the number of tens she had counted. This is consistent with using her right hand only (rather than both hands) to keep track of counting by ones.

Quotitive division involving a partially screened array (12÷2). On this task Tania first raised her 10 fingers simultaneously. During the next 13 seconds she alternately looked ahead and at her fingers several times. She then counted subvocally for 7 seconds while looking at her fingers, and finally looked up and answered "6". In the ensuing discussion Tania explained her strategy. A viable explanation of her solution consistent with her explanation is that she used her 10 fingers to symbolize 12, while realising that 12 involved two more than 10 which she could not show on her fingers. She then counted each pair of fingers from one to 5, and counted the additional pair that she could not show on her fingers.

Multiplication in written format. In attempting to solve "8X4" Tania first counted from "1" to "4" in coordination with sequentially raising four fingers on her right hand. On completing her count to "4" she raised one finger on her left hand. She then counted from "5" to "8" while raising the same four fingers on her right hand in turn. On completing her count to "8" she raised a second finger on her left hand. She continued in this vein making two more lots of four counts and raising the same four fingers in turn for each lot of four counts. She then answered "16". The interviewer told Tania that she had worked out 4 fours only. Tania then continued to count four more fours in the same vein and answered "32". Throughout her count her right hand served to keep track of the items in each four, and her left hand kept track of the number of fours. Prior to counting the eighth four for example, two fingers on her left hand were raised. After counting the eighth four she raised a third finger on her left hand. Thus is was clear that she could use the fingers on her left hand only, to symbolize a number in the range 1 to 5 or 6 to 10 as required. Finally, her differentiated use of the fingers on each hand was the same for multiplication as it was for addition and subtraction. In both cases she used her right hand to keep track of the number of ones and her left to keep track of the number of composites — tens in the case of addition or subtraction, and fours in the case of "8X4".

Helen

Addition involving a 1-digit addend. Helen counted-up-from to solve 19+3 and used her fingers to keep track. Her strategy was the same as that used by both Sally and Tania on this task. When solving this task Helen counted subvocally with her hands under the desk and after 2 seconds answered "22". In the ensuing discussion she demonstrated her strategy by saying "20, 21, 22" in coordination with sequentially raising three fingers.

Subtraction involving a 1-digit minuend. Helen solved 7-5 by first building a finger pattern for 7, then sequentially lowering the two raised fingers on her right hand, followed by three of the five raised fingers on her left. She then looked at her remaining two raised fingers and answered "7". Her strategy was the same as that used by Sally on this kind of task, and the same as one of the two strategies described by Tania for solving this task.

Addition involving two 2-digit addends. Helen solved 43+21 in the same way that Sally did, that is she counted from “44” to “64” and used the fingers of both hands to keep track. She solve 37+19 in similar vein with the exception that she counted from “20” to “56” rather than starting from “38”. As before she used the fingers of both hands to keep track. After she solved the task the interviewer asked how she knew to stop at “56”. She replied, “Because once you’ve done three tens you add a 7”. That she started from “20” is most likely attributable to confusion as a result of discussion immediately prior to her solution, during which she talked about “going by the lower number”. In a post-discussion she asserted that starting from “37” would be the quicker way. The interviewer then asked if there was a quicker way still, without going by ones all the time. After 8 seconds she replied “You could go by threes”.

Addition in vertical format. Approximately five minutes after solving 43+21 — written in horizontal format, the interviewer asked Helen to solve 43+21 but on this occasion wrote the task in standard vertical format. In solving this task Helen said, “well I would do umm —, three plus one is four (writes “4” in the appropriate place), and I would do four plus two (pauses briefly) is six (writes “6” in the appropriate place). Makes 64”.

Subtraction involving a two 2-digit minuend and a 2-digit subtrahend. Like Sally, Helen used “counting-down-from” in attempting to solve 67-38, and as in the case of the other tasks above, she used the fingers of both hands to keep track of her counts. She made 33 counts instead of 38 counts — that is she regarded 38 as three tens and three ones, and also omitted one number word, thus answering “33”. In an ensuing discussion Helen said, “I’ve done the tens and then an eight”. The interviewer asked, “How many tens?”, to which Helen answered, “three”.

Quotitive division involving a partially screened array (12÷2). Helen solved this task by counting from one to 12, in coordination with raising two fingers of her right hand in turn six times, and also raising a finger on her left hand for each pair of number words uttered. After uttering “12” she lowered the five fingers of her left hand and then raised her left thumb, saying “six”. Helen’s solution of this task was practically the same as Sally’s (see above). The only difference was Helen used her left hand in the way Sally used her right, and used her right hand in the way Sally used her left.

Multiplication in written format. In attempting to solve “8X4” Helen counted from “1” to “31”, and in doing so raised five fingers of her right hand and three fingers of her left, four times. She raised two fingers on her right hand in coordination with saying “12” and thus made 31 counts only. Again her strategy was practically the same as Sally’s.

Bruce

Addition involving a 1-digit addend. Bruce used an adding through 10 strategy to solve 19+3. When explaining his solution he said, “I take one from that (pointing at “3”) and put one on that (pointing at “19”) so that makes 20. And then I put two (pointing at “3”) on 20 (pointing at “19”), which is 22.

Subtraction involving a 1-digit minuend. Bruce explained his solution of 7-5 by saying, “Well I always know whatever that answer would be it would have to make 7 (pointing first to the right of “7-5” and then to the “5”). So I put 2 there (pointing to the right of “7-5”) because 2 plus 5 equals 7.

Addition involving two 2-digit addends. Bruce solved “43+21” by immediately saying “64”. The interviewer then asked, “Which ones did you add first?”, to which Bruce replied, “the ones column”. When attempting to solve “37+19”, Bruce looked at the card for 6 seconds and then said “46 —, no 56 —, no 46. I mean 56, — 56”. The interviewer asked Bruce how he had solved the task. Bruce replied, “because I add those two (points to “7” and “9”), and 7 plus 9 doesn’t equal —, it isn’t under 10, so I —, it equals 16, so I put the one there (points to the top and left of the “3” in “37”) and I put the 6 there (points below “37+19”), and 1 plus 3 plus 1 (points in turn to top left of “3”, “3” and “1” in “19”), equals 5”. The interviewer then asked, “How do you know 7 and 9? How do you work that out?”. Bruce replied, “I just add one onto there (points to “9”) and then what’s left I add on to 10”.

Subtraction involving a two 2-digit minuend and a 2-digit subtrahend. When attempting to solve “67-38” Bruce looked at the card for 6 seconds and then answered “31”. In explaining his answer Bruce said, “8 take away 7 equals 1 (points in turn to “8” and “7”), and 3 take away 6 equals 3 (points in turn to “3” and “6”)”.

Quotitive division involving a partially screened array (12÷2). Bruce solved this task by immediately answering “6”. He explained his answer by saying, “It’s easy. You just do 2 times 6 and that would equal 12”.

Multiplication in written format. When solving 8X4 Bruce looked ahead for 11 seconds and then answered “32”. He explained his answer by saying, “Well, I do 8 plus 8 plus 8 plus 8 equals 32.” When asked how he worked out 8 plus 8 plus 8 plus 8, Bruce answered, “I do 8 and another 8 which would equal 16, and then I take 4 from that 8 because that would make it 20, and there’s only 4 left over and that’s 24, and then I put 6 from the last 8 on that which would equal 30, and there’s 2 left and that would equal 32”.

Findings

As seen above, three of the four students all used one type of strategy on all of the tasks. Thus the close similarities in the strategies used by Sally, Tania and Helen were particularly striking. Not only did they use counting-by-ones using fingers to keep track in the cases involving a 1-digit addend or subtrahend but they also did this in the cases involving a 2-digit addend or subtrahend. Thus they did not increment or decrement by tens. Again, these students used counting by ones using fingers to keep track to solve multiplication and division tasks. Further, Tania’s differentiated use of each hand in multiplication — her right hand kept track of ones and her left kept track of the number of times she counted the repeated addend, seemed to interfere with her use of fingers to keep track of ten when solving 2-digit addition and subtraction. By way of contrast, Bruce used adding-through-ten to solve 19+3 and 8X4, and used 5+2=7 to justify his answer to 7-5 and 2X6=12 to justify his answer to 12÷2. As well, he used a kind of 1010 strategy (e.g. Beishuizen & Anghileri, 1998) to solve 43+21 and in his attempt to solve 67-38.

Conclusions and Implications

The cases of these students vividly illustrate Gray’s (1991) contrasting procedural and deductive approaches to simple arithmetic. The procedural approach is typical of below average students and “is dominated by the use of counting” (p. 569). The deductive approach is typical of average and even more so, of above average students and “makes use of other known knowledge” (p. 569). As stated earlier, this study was conducted in the context of refining SENA2 (an early number assessment schedule used in CMIT). Three important questions arising from this study are: how widespread amongst students in 3rd grade and above is the use of strategies like those used by the first three students described above? How likely are students who use counting by ones on addition and subtraction, to develop similar strategies for multiplication and division? And, how likely are students who routinely use these strategies, to encounter increasing difficulty and frustration with arithmetic as they progress through the middle and upper primary years and beyond?

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