Swaps and Switches: Students' Understandings of Commutativity

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This paper engages current thinking about the links between students' early experience with number properties and their future understandings of algebra. It reports on a study of students' understandings and explanations of commutativity. Central to the analysis are both the forms of representation students draw upon and the language they employ to make their understandings explicit. The investigation reveals that despite the rhetoric of educators and the intention of curriculum developers, student learning continues to remain at the procedural level, constraining rather than enabling algebraic reasoning.

"The power of mathematics", according to Deborah Ball (1993), "lies in part in its capacity to represent important relationships and patterns in ways that enable the knower to generalise, abstract, analyse, understand" (p. 163). Students who are successful in mathematics have moved beyond definitions, concepts and skills to an engagement with the underlying structure and regularities of mathematical operations, and are able to express their insights about mathematical relationships. The capacity to generalise, to construct ways of representation, and to offer justifying arguments and explanations, is a defining feature of a sophisticated mathematical experience.

Algebraic thinking relies upon insightful expression of the nuances between numbers and operations; these very same principles that govern symbolic representation and structure in algebra build on students' extensive experience with numbers and their properties. Indeed, as Schifter (1999) argues, implicit to the strategising efforts which students employ for calculating and for solving problems is an engagement with algebraic reasoning. For example, most students make implicit use of algebraic reasoning when they change the order of the numbers in their addition calculation. However, although students draw on the fundamental properties of addition, most are not aware of the underlying structure and properties of commutativity and hence are not able to make connections between the structure of algebra and the structural properties of systems of numbers (Carpenter, Franke, &, Levi, in press). Kaput (1999) and Schifter (1999) have both argued that many students fail to recognise that the preliminary ideas of algebra are merely the generalisation of arithmetic. Recognition of this link is crucial for "breaching the cognitive gap" (Warren & English, 2000, p. 624) and takes the student beyond competence with the traditional procedural missing-value problem into some of the most important algebraic understandings which have to do with equivalence and the transformation of variables.

Current thinking among mathematics educators and curriculum developers shifts the mathematical experiences of students from a procedural focus to active engagement with mathematical ideas, promoting a vision of teaching and learning opportunities which encourage reflection and argumentation (Yackel, 2001). Integral to our research are those forms of argumentation central to students' understanding of commutativity. We asked: What understandings do students have about the commutative property in relation to addition, subtraction and multiplication? Our theoretical framework is derived from Krummheuer's (1995) standpoint on argumentation. Krummheuer offers a reflexive perspective on learning, maintaining that a student's active involvement in a process of argumentation influences learning, and in turn, the student's process of argumentation is

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coursed through his or her cognitive capabilities. Through exploration of their own and other students' conjectures, students who engage in forms of mathematical argument evolve an understanding of what constitutes an appropriate explanation or justification. More importantly, they are better positioned to make explicit connections between arithmetic experiences and algebraic reasoning.

Research Study

This paper reports on data drawn from New Zealand's National Education Monitoring Project (NEMP). The latest available data, involving 3000 Year 4 (ages 8-9) and Year 8 (ages 12-13) students, assessed content knowledge and process skills in mathematics (Flockton & Crooks, 1997). The reported project focuses on students' knowledge of and articulation of ideas about commutativity .

The researchers viewed videotapes of 50 Year 4 students and 50 Year 8 students randomly selected from the NEMP bank of student responses. They investigated and analysed the differences in students' ability to recognise the commutative law and in their capacity to explain this law in everyday language. The videotapes provided a means to paint a national portrait of mathematics understandings presented verbally by students—the plain talk, the tangled persuasion and the resounding eloquence—at two distinct year levels. But the videos did more than this: they provided a visual image rich in descriptive power which allowed us to capture the processes involved in the demonstrated rationales offered by the students.

Each student was introduced to the assessment task by the interviewer's words: "Let's imagine that someone in your class needs some help with maths and that you are going to try to explain the answers. I'll ask the questions and you try to explain the answers. You will need to say more than 'yes' or 'no' because you want them to understand. Use the cubes to help show what you mean." The students were then asked the following questions:

- Is 4 plus 3 the same as 3 plus 4?
- What about 4 minus 3 and 3 minus 4? Are they are same?
- Does 2 times 5 give the same answer as 5 times 2?

Numeric representations on cards were placed on the table for each question. For example, the card 3 + 4 and the card 4 + 3 were both presented to the student for question one.

The researchers transcribed half of the tapes each then carried out a reliability check of the transcriptions of the other fifty. In this way both researchers viewed all 100 responses. Initial codes, listing all explanations and/or demonstrations which were explicit contrasts from other criteria, were refined as the researchers re-applied the system to the actual study tapes. The final choice of the best-fitting category for any response was by mutual agreement, arrived at through re-viewing the tape segment and/or further discussion.

Results and Discussion

The focus here on how students articulate their conjectures goes beyond a simplistic interest in communication engagement. If students are learning with understanding, they are engaged in justification. It was found that students *can* engage productively in justifying conjectures, yet many struggled to appeal to a model or use a language with the rigour and precision required to convey meaningful learning.

A Question of Addition

All but two students in the study volunteered that 4+3 is the same as 3+4. The arguments of these dissenting students (both from Y4) were based on differences that the students observed in the placement of the numbers on the two cards:

> No, this one here has 4+3, and this one here has 3+4.

All other students responded positively and were able to offer some explanation in support, or were able to justify by example. Responses were coded according to the following indicators:

- 1a The numbers are 'switched around'. (15 Y4 : 13 Y8)
- 1b Both sides add to seven or the same amount (11 Y4 : 11 Y8)
- 1c The same numbers *and* they both add to 7 or to the same amount. (19 Y4 : 25 Y8)
- 1d Other. (5 Y4: 1 Y8)

Explanations for 1a included the responses of students who either (i) switched around a model of *one* set of three and four cubes to represent the second card, or (ii) demonstrated the equivalence of the *two* spatial arrangements of cubes that modelled each respective card.

> Yes, you have 3 cubes and 4 cubes and it's just the same as if you have these cubes [refers to the 4 cube set] and put 4 cubes in front of the 3. (Y8)

Number switches provide a powerful visual image to the student. Schifter's (1999) research class coined the expression "turn-around" to convey the numeral switch in number. In the current study "turn-around" came in many other linguistic guises: "going that way, and the other is going the other way", "switched around", "put backwards", "in a different way", "the opposite way", "changed around", "upside down", "swapped around" and "changed sides".

It is possible within 1a coding that some students' justifications were based on an appeal to authority, involving the repetition of someone else's claim rather than on any meaningful understanding of commutativity. We make this suggestion knowing that a significant proportion (21%) of students justified the commutative property for subtraction on the basis of the same 'number switching' argument.

Likewise, it was unclear as to whether all students who provided explanations of type 1b were aware of commutative law. It is possible that in cases where students appeared to count each conjoined set separately, that the equivalence conjecture was backed by a similar warrant as might be applied to the equation 3+6 = 2+7.

> Yes, because they make the right answer. If you get 4 cubes and add 3 onto them they make 7 [counts all the cubes]. And if you get 3 cubes and add 4 onto them, that makes 7 [counts all the cubes]. (Y8)

> I've put the equals sign in both of them and they add up to the same one. (Y4)

The videos offered a wealth of information concerning differential levels of confidence and verbosity between Year 4 and Year 8 students. Implicit to the Year 4 students' characteristic long pauses and inflections in tone was an appeal for reassurance and affirmation. Their responses were often marked by unsophisticated and imprecise language:

> Um, yes, it is because that one [points to the card 4+3] is going that way and the other one is going the other way. So they both equal the same number. That equals 7 [points to 4+3] and so does that cos I counted up [counted up 3 and 4 cubes]. (Y4)

In comparison, a Year 8 student demonstrated the equality of the two addition statements in relation to the equal sets of cubes:

> Yes, because they both equal 7. 4 plus 3 is seven [joins 4 cubes and 3 cubes]. And that doesn't matter if you have them the other way [switches the cube sets] because they'll still be the same amount of cubes.

Analysis of the 1c coding responses revealed that more Year 8 students supported the notion of 'switching' by providing an additional warrant of a form similar in intent to: "you get the same answer", or "they both equal 7".

> Well, it can be because 4+3, you just get 4 cubes and 3 cubes and then you just swap around. And you just get the same answer because that there, you've got 3 plus 4 and [swaps the order of the cubes], you've got 4 plus 3. (Y8)

One student attempted to take the argument to a higher level of generalisation. However, in essence while wanting to provide an argument that is general, the student falls short, in that her argument simply restates the conjecture in slightly different words:

> Yes they are the same because they've got the same numbers. And in pluses it doesn't really matter how you have the numbers. They'll equal the same if the numbers are the same. (Y8)

While answering the given question was not conditional on supplying a generalisation, it was surprising, given the centrality of commutativity to the development of early number understanding, that generalised statements were not offered by more students. When we consider alongside this observation that some Year 4 students could not provide a model or explanation to support their conjecture, we are led to wonder if discussions about properties of numbers and operations have been commonplace in the New Zealand classroom.

A Question of Subtraction

Coding indicators for subtraction:

- 2a Yes they are the same. (14 Y4 : 3 Y8)
- 2b 4-3=4, and 3-4=3 (6 Y4 : 0 Y8)
- 2c Double model (2 Y4 : 10 Y8)
- 2d Correct modelling & not explained; or no model & explained (9 Y4 : 5 Y8)
- 2e Comparison (1 Y4 : 2 Y8)
- 2f Correct modelling and uses 'minus' or 'you can't' (18 Y4 : 30 Y8)

Students in the 2a category backed their argument by making reference to the identical numbers on the two cards. A few students appeared to be unsure of how to read a subtraction problem, claiming that the problems would have to be the same because on the second card the numbers were *"Just swapped round"*.

> Yes, you've got 4 there and 4 there. And if you take away. And you get the same answer. (Y4)

> Yes, they both have the same, but swapped around and equal to one because, like, if you had 4 minus 3 it would be 4 blocks and take away 3 and you've only got one left. And 3 minus 4 is just the same as 4 minus 3 but it's round the other way. You've got 4 blocks and you just take away 3 blocks again. (Y8)

Some students provided a further warrant by arguing "*It's just like the other one*" [the addition problem].

> They're the same as well. Except they're minus so they both equal the same number. Exactly like the other one but they're minus. And they're the same, by two 3's and the same by two 4's. (Y4)

Analysis revealed that 13 of the 38 students, who initially supported the conjecture that 4-3 was the same as 3-4, altered their response as they proceeded with an explanation or demonstration with the cubes. Proportionately more Year 8 students revised their assessment of commutativity for subtraction when prompted to provide a justification.

A consideration of the responses of the students who disagreed with the conjecture that 3-4 was the same as 4-3 (code 2b) raised an important issue concerning forms of representation which students gave to the problems. A large number of students (both Year 4 and 8) modelled, with the cubes, the numbers on the cards and then became confused about what exactly to subtract.

> You've got 4 then you take away 3 and that just leaves 4 [i.e., □□□□ – □□□ = □□□]. And then you have 3 and you take away 4, you'd leave yourself 3 [i.e., □□□ – □□□□ = □□□]. (Y4)

Others (code 2c) who modelled the numbers on the card [4-3], self-corrected by taking away 3 cubes twice to offer '1' as the answer. The frequency of these mis-representations suggests that modelling basic subtraction problems was not firmly established in the students' experiences. The inability to directly model such elementary problems with concrete materials will invariably hinder the student's propensity for visualisation and abstraction deemed necessary for algebraic thinking. In earlier studies, children's reluctance or inability to translate between the written and concrete representations using a model depicting 'separation' led Hughes (1986) to conclude that "some of the children's responses revealed a major gap between their concrete and written understanding" (p. 132)—a conclusion that appears to be valid even today.

Not surprisingly, given no context for the subtraction problem, very few students chose to use a compare model to demonstrate their conjecture. Included here is one of the few demonstrated comparison examples, recorded in the 2e category:

> 3, you have to put one on. That equals 4 [constructs a column of 3 cubes alongside a column of 4 cubes]. So that's one left over. And if you do 3-4, 4 doesn't go and if you go, count backwards, minus 1 from zero, so it's minus one. (Y8)

One Year 4 student couched her compare explanation within a context meaningful to her:

> Yes, well pretend you've got 4 apples and you had 3 visitors come and one of them was you. You would give, um, and you had, you would give the person one apple each. And then 4 take away 3 would be 1. Because there's not enough for it to equal. See, there's a person and there's an apple and that's a person and there's an apple and you can see that's really 2 apples going into one more person. And you gave them the apples. And that's for you. And you've got one more apple left.

For those students (2f) who provided a clear demonstration of their understanding of the subtraction process, most stated that 4 minus three "can't be done"; sometimes offering an additional warrant that there were not enough cubes to take away:

> You can't do it because 4 is higher than 3. (Y4)

When confronted with a numerical situation that could not be contained within their number experience, some students volunteered that the answer to 3-4 would be zero or nothing:

You can't have 3 and take away 4 cos there's only one there. Take away 3 there. Take away 3. Nothing. (Y4)

Other students offered some explanation related to a negative answer for 4-3.

[For 3-4] Imagine you had 3 apples and somebody took, somebody came one night and took 4 out, out of 3. And then you would have, you'd have those 3 taken, and you'd have no more left. And that would be minus one. (Y4)

...but if you had 3 cubes and tried to take away 4 you would take away all your 3 and you won't have another one to give away so you would be minus 1. (Y8)

A Question of Multiplication

Coding indicators for Multiplication:

- 3a Models numbers on cards and says '7'. (9 Y4 : 2 Y8)
- 3b Models numbers on cards and says '10'. (5 Y4 : 2 Y8)
- 3c Known fact with no demonstration. (2 Y4 : 0 Y8)
- 3d Models correctly one set only, with explanation. (8 Y4 : 7 Y8)
- 3e Forms 2 sets of 5 cubes then rearranges to form 5 sets of 2. (11 Y4 : 28 Y8)
- 3f Forms 2 unique sets; one representing 5x2, the other 2x5. (8 Y4 : 6 Y8)
- 3g Other, including reverse modelling (7 Y4 : 5 Y8)

Demonstrations of the 3a kind based on a direct model of the numbers on the given cards do not constitute a legitimate representation of the commutativity of multiplication.

[Pointing to the card 2 x 5] You've got 2 on this one and you add 5 and you get 7. [Referring to the card 5x2] You've got 5 and you just add on 2 and it equals the same. (Y4)

Likewise, students' warrants categorised under 3b ignore the array structure of multiplication. Instead, students use a known fact to justify the 'sameness' of 2 x 5 and 5x2.

I: You've made a set of 5 and a set of 2. [Y4 modelling the cubes □□□□□ □□]

S: [reaffirming by counting] 1,2,3,4,5. And a set of 2.

I: How much does it come to?

S: 10. The same as that one [pointing to the card 2x5]. 10.

Analysis revealed that 28% of Year 4 students recited a known fact and modelled unconventionally. 8% of Year 8 students began by modelling the numbers on the cards then set aside this initial representation and started afresh:

They're the same. It's like the other one [referring to addition problem; models the numbers 2, 5 □□ □□□□□]. There's 2 5s □□□□□ □□□□□ [changing arrangement of cubes]. There's 5 plus 5 there. Two 5s equal 10.Then 5 pairs of 2 [□□ □□ □□ □□ □□] So 2, 4, 6, 8,10. Same thing. (Y8)

Student responses denoted as 3c begin with an appeal to authority and when pushed for further clarification are unable (or unwilling) to provide a model, offering instead an approach incorporating repeated addition:

Yes it will [equal the same]. The order is just changed around. You get charts. If you don't know the times table. You get charts to name them. It's just 5 times 2 is 10. All you're doing is just doubling, tripling, 5 times 2 is 10. It's just the same as plus. You're going up 2 or three numbers [pause]. 5

> ones are 5 and you just add on another 5 and that makes 10. You go up 5 each time. 5 twos are 10. Just the same, It's like 2 times tables. You use the 2 and you go up 5 times. (Y8)

Additive reasoning develops intuitively from everyday additive encounters (Lamon, 1995; Simon & Blume, 1994). Multiplicative reasoning, on the other hand, is a more elusive development which requires progressive pedagogical exposure to multiplicative structures. Many researchers (e.g., Sowder et al., 1998; Thompson & Thompson, 1994) have noted the conceptual difficulties associated with multiplicative reasoning but insist that such experience is important not only for students' conceptual understanding of rational numbers but crucial to the development of the ideas fundamental to algebra and calculus. Whilst additive thinking, in contrast, is "a familiar and comfortable means of solving multiplicative problems" (Jacob & Willis, 2001, p. 306), it is also limiting.

In the following examples a progressive development from additive reasoning and tending towards multiplicative thinking is made explicit:

> Yes [they are the same] because 2 5s [□□□□□ □□□□□] is the same as 5 times 2 because you put 2 [□□], 2[□□ ,] 2[□□], 2[□□], 2[□□]. It's 10. Then either one is the same. 10. (Y4)

> You get 5 times 2 is 10. And 2 times 5 is [counting out 10, □□□□□□□□□□] So that's 2 times 5. This is a hard one, 2 times 5 is 10. So 5 times 2 is 10 as well. ...if you had 5 times 1, it would just be 5. So 5 times 2 is adding another one on for every one. That's 10. That's the same. (Y4)

> Well, you've got...you start off with the 5 times 2. You'd need to do 2 sets of 5. So you'd have [□□□□□ □□□□□]. 5 sets of 2 with other blocks [□□ □□ □□ □□ □□]. Once I'd done that I'd say, OK, count how many blocks there are. And they'd say it's the same because once again, it's the same numbers but just a different method of showing it. And what it is...you've got 2 sets of 5 that add up to 10 and 5 sets of 2. And you're doing the same thing except that you're doing 2 of these [pointing to the card 2x5] and 5 of these [pointing to the card 5x2]. (Y8)

Responses classified in the 2f category either exhibited confused thinking or reversed modelling. Imaging by reversal appeared to make a good deal of sense to some students:

> Yes, cos you've got going 2x5 [□□ □□ □□ □□ □□]. It's got, you've got 2, and you've got times 5. It's like. Um, it's the same cos you've got to go 2 times 5. You've got all these [pointing to cubes] and you just, so you've got 2, you've got 5 lots of 2, so you can either add them together or if you know your tables you can go 2x5, and it's the same with the other one. It's just, you're going, you've got 5 lots of 2 [□□□□□ □□□□□].

It is our contention that few of the research participants exhibited sophisticated multiplicative reasoning and commutative understanding. A student with multiplicative thinking and commutative understanding operates at a higher level of abstraction and would probably reveal that thinking by *implicitly* reorganising the array structure. Hiebert and Behr (1988) argue that, whereas addition operates on singleton units, multiplication deals with composites. An array enables the student to see, simultaneously, the singleton units and the aggregation of those units. None of the sampled students appeared to construct and coordinate composite units through an array—visualising the array in two unique orientations—that would be suggestive of reasoning multiplicatively and thinking in commutative terms.

Conclusion

Justification is central to mathematics. In order to make sense of and genuinely understand mathematical concepts and move beyond solutions at the surface procedural level students need to be able to explain and justify and convince themselves and others.

Reasoning through argumentation works on many levels of sophistication from appeal to authority, justifying by example, analogising and generalising. In the process of learning those forms of argumentation which are persuasive and useful to the problem in hand students acquire a firm foundation for the learning of algebra.

Descriptions from this study indicate that, despite the intent of curriculum developers, many students have learned arithmetic in a way which is not conducive to the enrichment of structural understanding. We maintain that the focus on procedural learning which persists has little flow-on effects for the development of algebraic reasoning. To foster this development we suggest that students need more productive modes of instruction than are currently in operation. Forms of conjectorising and generalising about the properties of numbers need to be encouraged. Students need opportunities to make explicit their understanding of *why* number properties such as commutativity 'hold good'. And they need to be able to apply these procedures in a variety of contexts. In subscribing to these approaches teachers can ensure that students will learn to make the connections between the students' own understandings of number properties and the study of algebra.

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