One Student's Understanding of the Concept of Function

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This study aimed to investigate the ways in which one student developed his understanding of concept of function and the use of his concept image of function when solving problems. This student's responses from a function concept questionnaire, a function test and interviews were classified into one of three categories: action, process or object concept of function.

It is widely accepted by mathematics educators and teachers of mathematics at all levels that the concept of a function plays an important role in the algebra curriculum. The improvement of instructional strategies to promote higher levels of student understanding of the function concept is a key goal of the secondary school mathematics curriculum. Researchers including Breidenbach, Dubinsky, Hawks and Nichols (1992) and Sfard (1991) have attempted to find better ways of building students' understanding of the function concept. It has been suggested that using a graphing calculator and a computer can assist students to construct a richer understanding of the function concept (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Hollar, 1996; O'Callaghan, 1998; Slavit, 1994).

This study is part of a wider study of the introduction of graphing calculators to Thai secondary mathematics students. The study investigates the ways in which the students' develop the concept of function and describes their learning in terms of the "action-process-object" framework (Asiala, Brown, De Vries, Dubinsky, Matthews & Thomas, 1996).

The Concept of Function

Vinner and Dreyfus (1989) classified the mathematical concept of function into two distinct aspects, concept definition and concept image. A concept definition is a formal mathematical definition of the concept. It is directly introduced to the students in words of phases. On the other hand, a concept image is a set of all the mental pictures, and a set of all the properties associated in the student's mind with the concept name. The concept image includes the visual representations, experiences and impression evoked by the concept name (Thompson, 1994). Vinner and Dreyfus (1989) stated that the student's concept image results from experiences with the concept, and is not necessarily the same as the concept described by the definition. In order to handle the concept, according to Vinner (1983), a student needs a concept image rather than a concept definition. Therefore, it is important to investigate the student's images of mathematical concepts. This study supports this claim by exploring a student's concept image of the concept of function.

Dubinsky and his colleagues (Asiala et al., 1996; Dubinsky, 2000) proposed the APOS theory developed from Piaget's idea of reflective abstraction. In the APOS theory, the construction of a mathematical concept begins with manipulating previously constructed mental or physical objects to form *actions*. An action is a transformation of mental or physical objects and the individual can carry out the transformation only by reacting to

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external cues that give precise details of the steps to take. A student holding an action concept is able to calculate values of a function for specific values from the domain, or is able to find specific points on a graph representing the function.

For an interiorized *process*, an individual reflects on the action and constructs an internal operation that the individual is able to describe, or reflect on, as the steps in the transformation without necessarily performing them. At this stage, a student is able to think of function as receiving one or more inputs, or values of the independent variable, perform one or more operations on the inputs and return the results as outputs or values of the dependent variable. The student also thinks of the function as a formula, graph, table of values, mapping diagram, or function machine. The constructed process can be reversed or it can be coordinated with other processes to form a new process.

An *object* is constructed through the encapsulation of a process. A student becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such a transformation. As an object, function is thought of as the correspondence between two variables. The student can de-encapsulate the object back to the process concept in order to manipulate it to form a new process. In this stage, the student also thinks of a function as something to which actions or processes may be applied. Finally, the actions, processes and objects can be organized in *schema.* According to Vinner and Dreyfus (1989), a concept image comprises the set of all mental pictures associated in the student's mind with the concept name, and all of the properties related to them. Moschkovich, Schoenfeld and Arcavi (1993), and Slavit (1994) stated that a student who understands the function concept as an object can use a function and its properties in solving a novel problem. Slavit (1994) also stated that a student who perceives the function concept as an object has the ability to realize the equivalence of procedures that exist in different representations and to generalize procedures across function classes.

It is possible that a student may think about a function in one manner, but uses a different concept of function to solve problems (Slavit, 1994). Slavit stated that a "student may talk in general about function as a relation between two sets, but that same student may think of only specific functional properties when problem solving" (p. 50). This implies that investigating students' thinking related to the concept of function should incorporate a range of situations and activities to allow for the display of the full range of thinking.

Methodology

The study reported in this paper investigated one student from a wider study of 13 year ten (approximate age of 15-16 years) students who studied in the first semester of academic year 2001 in a Thai secondary school. The study was not part of the regular teaching program and was conducted after school time. The researcher taught the class. The teaching experiment consisted of a series of teaching episodes over 7 weeks, with 2 one and a half-hour episodes per week. The following topics were addressed in the course: introduction to a graphing calculator, relation, function, linear function and linear equation. A constructivist approach incorporating peer collaboration and the use of a graphing calculator was applied. Within each teaching episode, the students worked on the learning tasks in groups of two or three. They performed the tasks by discussing and sharing information with their group-mate. In each session, some groups were asked to present their answers to the class using an overhead projector. At the end of the study, a function concept questionnaire and a function test were administered, and three students were interviewed. The interviews were videotaped. The investigation reported here focuses on one of the three students interviewed, namely "Piya". Piya complete year 9 with a grade point average 3.43 (4 point-scale) and received "A" in the mathematics course. He had no experience in using a graphing calculator either in and out of school and he had not been introduced to the function concept in his earlier mathematics studies.

Two written instruments were used to gather data. The function concept questionnaire consisted of two parts. The first focused on students' concept definition of function as well as asking for one example of a function in a practical situation. The second part focused on students' concept image of function, having them classify various relations given as either algebraic equations, or tables of values, or graphs or sets of ordered pairs. Students' responses were elaborated on in the interviews.

The function test was developed from tests used by O'Callaghan (1998), Hollar (1996), Eisenberg and Dreyfus (1994), and Slavit (1994). The test was designed to assess each of the following aspects of the conceptual knowledge of function: interpreting functions (5 questions); modelling functional problem situations (4 questions); and transforming between different representations and within the same representation of a function (4 questions). Students' responses were also elaborated on in the interviews. The interviews were conducted on the next two days following the test.

Based on the APOS theoretical framework, all student's responses from the function concept questionnaire, the function test and the interviews were classified into one of three categories: action concept, process concept or object concept. The responses of the one student who was the focus of the study reported here were classified and then an overview of the student's concept of function was constructed.

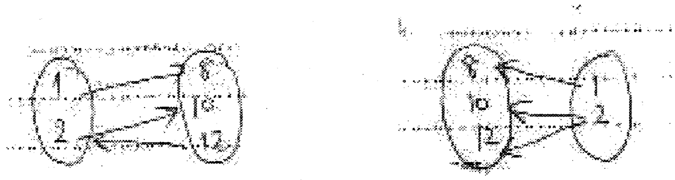
Results

Concept Definition and Concept Image of Function

The student's responses from the function concept questionnaire were used to explore the student's concept definition and concept image of function during the interview.

The student, named Piya, gave the definition of a function as:

> Two sets of numbers map with each other, such that for every element in the first set maps only one time with an element of the second set, but each element in the second set can be mapped with elements in the first set more than one time.



Not a function A function

Figure 1. Piya's examples of a function.

Piya's response indicated that he held an *object* concept definition of function. He thought of a function as a relationship between two sets of numbers, a mapping, with a univalent property. His concept definition of function was limited to the mapping between two sets of real numbers as he was unable to expand on this definition in the interview. He gave two examples (Figure 1) using mapping diagrams to support his definition, but the directions of the arrow lines were mixed. In the first example, he used an arrow line directed from 12 to 2. It appears that he intended to show that the number 2 mapped with 10, and if 12 was also mapped with 2, then the mapping diagram did not represent a function. In his explanation he stated: "...2 with 10 and it cannot map with 12, one number of the first set cannot map with two numbers of the second set". As in the first diagram, Piya used arrow lines in the wrong direction for the second mapping diagram. He explained: "1 maps 8, but one element of the second set, 2, can map with elements of the first set more than once, because from the definition each element of the first set can map with only one element of the second set". He appears to be using a learned definition but is confused about the meaning of a "many to one" relationship.

Piya's answers to the second part of the function concept questionnaire involving the identification of functions showed that he believed that parabolas, represented by either a graph or an equation were not functions because they are not one-to-one. When talking about the graph of a parabola, Piya checked first with a vertical line and then checked again with a horizontal line.

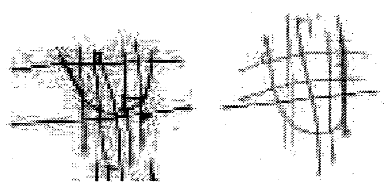


Figure 2. Piya's working on parabolas.

TR (Teacher researcher): Why do you answer that $y = x^2-2x+5$ is not a function?
P: *Its graph is a parabola. The parabola makes two points.* These points are the points that the horizontal line crosses the graph as the picture. (Later) I think all parabolas are not functions. Although, a vertical line passes the graph at only one point, a horizontal line passes the graph at more than one point.

The one-to-one relationships belief was also apparent in Piya's explanation of whether a relationship expressed in a table was a function, many-to-one relationships being classified as not functions. However, Piya classified a discontinuous graph with a split domain (Figure 3) as a function using the vertical line test even though it was a many-to-one relationship. For other non-linear graphs, Piya applied the vertical line test first. If vertical lines met the graph in only one point, he then applied a horizontal line test as he had for a parabola.

$$y=\begin{cases}0; & x<1\\ x; & 1\le x\le 3\\ 1-x; & x>3\end{cases}$$

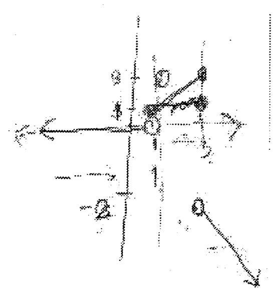


Figure 3. Piya's use of the vertical line test

Concept Image of Function Used in Solving a Problem

Interpreting a problem situation. Piya had not been exposed to the composition of functions during the instructional period. He demonstrated his ability to master a problem situation involving the composition of functions. His work and transcript of interview were as follows.

Problem: Let $f(x) = 4x-3$ and $g(x) = x^2-x$, find the followings:

1) $f(-3)\times g(1)$ 2) $f(g(2))$ 3) $g(f(4))$ 4) $f(g(x))$

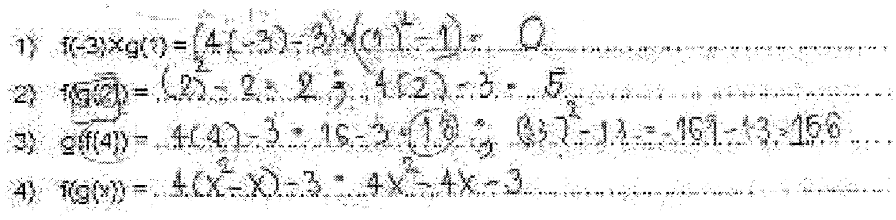


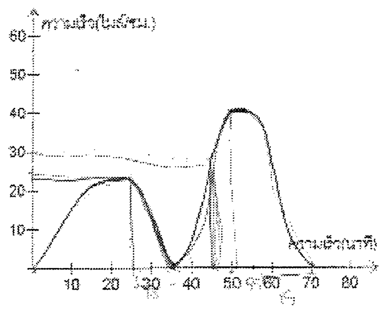
Figure 4. Piya's work on composition of functions.

P: To find $f(-3) \times g(1)$, firstly find $f(-3)$ by substituting $x = -3$ in f and secondly, find $g(1)$ by substituting $x = 1$ in g. Then $f(-3) \times g(1)= [4(-3)-3] \times[(-1)^2-1] = 0$. Because the second term equals zero (0). To find $f(g(2))$, find $g(2)$ first, we get 2, and substitute $x = 2$ in the equation of f and get 5. To find $g(f(4))$, we find $f(4)$ first and get 13. Then find a value of g by substituting $x = 13$ in the equation of g and we get $g(f(4)) = 156$.

Piya exhibited his ability to objectify his understanding of the function concept. He thought of the function as leading to actions or process, for example calculating $f(4) = 13$ and $g(f(4)) = g(13) = 156$. Later in the test and interview, Piya manipulated the perceived object to find a new object when he substituted $g(x) = x^2 - x$ in $f(g(x))$ and simplified. That is, he created a new process (composition of functions) which required an object view of function.

The following problem involved interpreting a function represented by a graph.

Problem: The graph below gives the speed of a cyclist on his daily training ride. During his ride, he must climb a hill where he pauses for a drink of water before descending. Use this graph to answer the following questions as accurately as possible.



1. Find the time when speed equals 30 mph.
2. During what time intervals was the speed increasing?
3. During which 10-minute interval did the speed decrease most?
4. When was the cyclist at the top of the hill?

Figure 5. Piya's work on a v-t graph.

Piya demonstrated his process and object conceptions of function when solving question 3, 4 and 5. A misconception was shown in interpreting the graph for question 5.

TR: Great. For question 3 your answer is the time interval 0 to 20 minutes and 35 to 50 minutes. Why?

P: Within the interval 20 to 25 minutes, the speed is almost constant.

TR: For the question 4, why do you think that the three intervals--20 to 30, 30 to 40 and 60 to 70 are the intervals that the speed decreased most?

P: Can I change my answer? The interval that the speed decreased most is the interval 55 to 65. When compared with the interval 25 to 35 minutes that the speed is increasing.

TR: When is a cyclist at the top of the hill?
P: From the graph, in the interval 50 to 60 minutes the cyclist is at the top of the hill.

Working with question 3, Piya demonstrated a process concept of function. He was able to interpret the graph in terms of interval reading. He could find the time interval that the speed increased. While doing question 4, Piya coordinated two processes, speed during the times 25 to 35 and 55 to 65 minutes in which the speed decreased, and by comparing a changing rate of the speed between the two intervals, chose the interval that the speed increased most, the interval 55 to 65 minutes. In question 5, Piya displayed his object understanding of function. He considered all points or all features of the graph (whole graph reading), and answered that in the interval 50 to 60 minutes the cyclist is at the top of the hill. It was an incorrect answer. His misconception was that the graph represented the picture of the hill and that the maximum point of the graph represented the top of the hill. He did not link the given information from the situation that the cyclist stopped riding for a drink of water at the top of the hill. That was, at the top of the hill the speed of the cyclist was zero miles per hour.

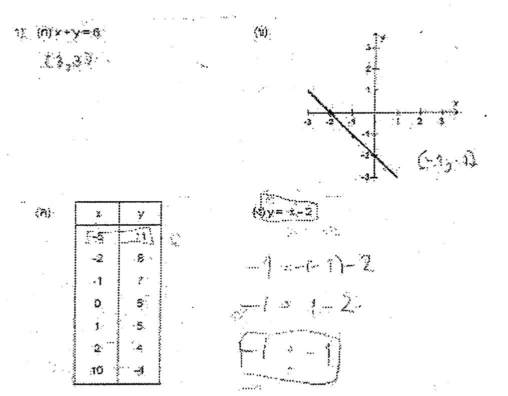
Modelling the problem situation. The following problem required students to model the situation with equations.

Problem: On your new job, you can be paid in one of the two ways.
Plan A: A salary of 50,000 baht.
Plan B: A salary of 35,000 baht plus a commission of 5% on gross sales.
For what gross sales is plan B better than plan A?

From his work, Piya displayed his understanding of function as an object. He knew that the problem situation could be represented with linear equations. He assigned a variable x as a monthly salary, and y as gross sales and constructed equations representing the relationships between the two variables for plan A and plan B. In order to find the amount of gross sales making plan B better than plan A, he then created an equation representing the situation by setting gross sales for plan A equal to gross sales for plan B and solved for x. It indicated that Piya viewed functions in the form of equations as objects on which he could operate. Unfortunately, he obtained the wrong answer. Mistakes were made in the process of formulating equations representing the problem situation. Instead of setting the amount of gross sales for x and the monthly salary for y, he set the monthly salary as x and the gross sales as y ($y = 50000x$ for Plan A and $y = 35000x - 20$ for Plan B). When asked for reasons, Piya could not explain why. He said, "I have no ideas about these".

Translating between different functional representations. Problems involving translating between different functional representations are usually solved in one of two ways, using a local strategy or a global strategy. The local translation strategy, or point-wise translation strategy, involves the student in checking several values of the independent variable in each representation and determining if they produce the same values of the dependent variable. This approach corresponds to a process view of function. The global translation strategy, an object view of function, involved the student using function properties and behaviours in the processes of translating. For example, when translating between two representations of a linear function, the student realizes that each has the same slope, and then checks the y-intercept or any other point. In the interview, Piya demonstrated a view of function as a process concept. He used only the local point-checking strategy when translating between different representations. The following are some of Piya's comments during the interview.

Problem: Determine which of the following functions are the same (Figure 7).



TR: Do you think functions in (a) and (c) are the same?
P: Compare the table in (c) with the expression in (a), you see, sum of each pair of x and y in the table is equal to 6. That is, data in the table satisfy the equation $x+y = 6$.
TR: Why are the graph in (b) and equation $y = -x - 2$ in (d) equal?
P: From (d), a graph of the function $y = -x - 2$ has the same point that the graph crosses the x-axis--for $y = 0$ we get $x = -2$, as the graph in (b). Let's check a point (-1, -1) on the graph in (b). The point (-1, -1) satisfies the equation $y = -x - 2$ in (d). Then the function in (b) is equal to the function in (d).

Figure 7. Piya's work during the interview.

In comparing the function representations, Piya called on his process view of function. However, when asked for an explanation of the general linear function $y = mx + b$, he displayed his object concept understanding of function by stating:

> The constant m represents the slope of a straight line. If m is positive, the line makes an acute angle against the x-axis. If the constant m is negative, the line makes an obtuse angle against the x-axis....[The constant b] tells the y-intercept of the graph. That is a point (0,b). For example, an equation $y = 5x+6$ passes a point (0,6).

Discussion and Conclusion

This study has provided an insight into one student's concept definition and the concept image of function. Vinner and Dreyfus (1989) stated that a student's concept image used in problem situations is not necessary the same as the concept determined by the definition. Students may discuss and think of functions in one manner, and use different conceptions of function to solve problems. Piya's concept definition of function was interpreted as an object concept. He defined the function as a relationship (mapping) between members of two sets of numbers under a univalent property. On the other hand, Piya used a variety of concept images to solve the given problem situations. In an identification of function task, Piya's concept image of function included it being a one-to-one relationship in some cases and a relation with a univalent property in other cases. Discontinuous graphs, graphs for which equations could not be specified, and equations with a split domain could be considered as function if they satisfied one of his concept images of function. In the interpretation of problem situations, he thought of a function as something to which actions and processes could be applied, and the properties and behaviours of the function were considered. In the modelling task, Piya displayed an object concept of function. He viewed functions as objects, which could be compared and operated on. Within the translation tasks, Piya used a local translation strategy, the point-checking strategy, to translate between different functional representations using a process concept. While Piya displayed a wide range of concept images and properties involving actions, processes and objects related to the function concept, his schema was at this stage immature. He was confused about a number of key ideas but his responses in the interview indicated that he was able to resolve those confusions with a little help from the teacher/researcher.

The study also indicated that perceiving a process or object concept of function did not guarantee the student's success in a problem solving situations. Success required the student to link perceived concept images of function with related processes needed in the problem solving situations. As Hiebert and Carpenter (1992) stated "understanding involves recognising relationships between pieces of information" (p. 67). Further research could investigate students' ways of using the concept of function in a wider range of problem solving situations.

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