The Role of Structure in Children's Development of Multiplicative Reasoning

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Key features of developing multiplicative structure were analysed from case studies of twenty-four students representing extremes in mathematical ability. Data drawn from a longitudinal study from Years 2 through 5 of schooling indicated that low ability students represented multiplicative situations without structure and development progressed from the use of pictorial to ikonic representations. Absence of any underlying structures persisted through to the end of Year 5 for half of these students. From the outset in Year 2, high ability students used notational representations with well-developed structures, and dynamic imagery featured strongly in their responses.

Multiplicative reasoning is essential in the development of concepts and processes such as ratio and proportion, area and volume, probability and data analysis. It is clear that failure to develop multiplicative structures in the early years impedes the general mathematical development of students into the secondary school, for example, in using algebra, functions and graphs. It appears that difficulties faced by older students can be attributed, at least in part, to the lack of development of an equal-grouping structure in early concept formation (Mulligan & Mitchelmore, 1997). Often young students' own representations lack any recognisable cohesive structure and they are unable to use their representations flexibly. Using structure is also important in the organization and interpretation of multiplicative situations shown as models, diagrams, tables and graphs. It is still unclear how the development of underlying structures, whether they are essentially mathematical or related to spatial organisation, influences mathematical development. This paper reports one aspect of a 4-year longitudinal study on children's number concepts. It describes the development of a theoretical framework for analysing the role of structure in multiplicative reasoning supported by data from twenty-four case studies of students followed from Years 2 through 5 of schooling.

An Overview of Research

Children need to develop and recognise underlying mathematical structure in order to understand how the number system is organised and ordered by grouping in tens, and how equal groups form the basis of multiplication and division concepts (Boulton-Lewis, 1998; English, 1999; McClain & Bowers, 2000; Mulligan & Mitchelmore, 1997; Mulligan & Watson, 1998; Sullivan, Clarke, Cheesman, & Mulligan, 2001; Willis, 2000). Research in early number development has identified counting, subitizing, grouping, partitioning and sharing as essential elements of multiplicative structure (Mulligan, in press). Structure can be identified in a variety of ways, such as by finding patterns of five dots in an array of twenty-five items rather than seeing twenty-five individual items. Children's early multiplication and division knowledge results from cognitive reorganisations of their counting, addition and subtraction strategies, and builds on number word sequences, combining and partitioning. The development of multiplication and division knowledge is described in order of increasing sophistication from initial grouping and perceptual counting to abstract composite units and repeated addition and

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combining and partitioning. The development of multiplication and division knowledge is described in order of increasing sophistication from initial grouping and perceptual counting to abstract composite units and repeated addition and subtraction, and to multiplication and division as operations (Mulligan & Wright, 2000).

Studies investigating the role of imagery in children's counting and numeration have also identified structural development of the number system from elements such as grouping, regrouping, partitioning and patterning found within the recordings of the numbers 1 to 100 (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, in press). Emerging structure is typified by numerals organised in a counting sequence, recorded continuously in a horizontal, vertical, curved or spiral formation. Mathematically gifted students' images have recognisable mathematical and spatial structure and use dynamic imagery (changing or moving images), whereas low achieving students show no signs of underlying structure. Students showing evidence of a more developed mathematical structure record accurate counting patterns using multiples (e.g., 3, 6, 9...), and marks or pictures showing a ten by ten (10 x 10) array structure. For example, Mellissa, aged 7 years (Fig 1) gave evidence of a high level of structure in her representation of 100 with her drawing of ten cubes. Robert, aged 9 years (Fig 2) drew a square and subdivided rows of separate squares, each square not being aligned to adjacent squares, and then recorded numerals for the numbers in squares, 1 to 17 being in the first row. It can be inferred that Robert has a much weaker understanding of the number system, as evidenced by his lack of structure, than Melissa.

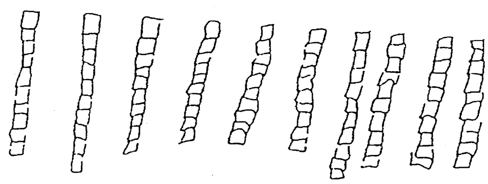

Figure 1. Melissa's image of 100.

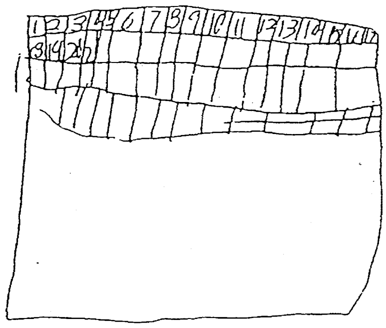

Figure 2. Robert's image of 1 to 100.

Mathematical structure has also been described in terms of students' spatial organisation of collections of two-dimensional and three-dimensional objects such as arrays of squares in rectangles and cubes in rectangular boxes (Battista, 1999a, 1999b; Battista, M. T., Clements, D. H., Arnoff, J. Battista, K. & Borrow, C., 1998). Other studies have investigated students' lack of structure in measurement of rectangular arrays, squares and other two-dimensional objects (Outhred & Mitchelmore, 2000; Reynolds & Wheatley, 1996). One of the difficulties is that students may not construct the row-by-column structure that identifies how squares fit together in a rectangle. Another problem is the underlying equal-groups structure required for counting rows and layers in multiples.

Background to the Study

The use of mathematical structure was highlighted in a longitudinal study of young children's intuitive models for multiplication and division. Mulligan and Mitchelmore (1997) found that the intuitive model employed to solve a particular problem did not

Observed Learning Outcomes model (SOLO) (Biggs & Collis, 1991). The analysis showed that the early formation of visual images in the prestructural and unistructural levels in the ikonic mode was critical to development of the equal-grouping structure of multiplication. Images initially lacking in structure at the prestructural level became more organised mathematical elements in the ikonic mode, i.e. random drawings of ikons began to show the structure of groups. Children initially giving prestructural responses became less reliant on physical models and idiosyncratic images and began to focus on numerical aspects of the problem in the ikonic mode. This development influenced their ability to interpret the semantic structure of the problem and their ability to represent equal-sized groups through physical or concrete models.

Children's Representations of Multiplicative Structures: A Longitudinal Study

Mulligan & Mitchelmore extended their research on intuitive models in a 3-year longitudinal study of second graders' representations of numerical situations involving counting, grouping, place value and multiplicative reasoning (Mulligan, Mitchelmore, Outhred & Russell, 1997). This project advances prior research through a longitudinal study integrating three key elements:

- a notion of mathematical structure commonly described as equal grouping, partitioning, or patterning inherent in multiplicative reasoning (Mulligan & Mitchelmore, 1997);
- the use of spatial structuring in mathematical representations which pertain to organisational features such as arrays and charts (Battista, 1999b); and
- the role of imagery in relation to developing both mathematical and spatial structure (Thomas et al., in press).

Analysis of children's drawings, symbols and explanations of their representations identified how children imposed structure, or lack thereof, on numerical situations. Low achievers were more likely to produce poorly organised, pictorial and ikonic representations that were lacking in structure. *These children lacked flexibility in their thinking; they were only able to replicate models of groups, arrays or patterns that had been produced by others.* Poor performance in multiplicative tasks was attributed not just to a lack of underlying equal-groups structure, but also to the primitive idea that unitary counting can be used to solve everything. Poor performance on simple ratio tasks was also linked to children's inability to visualise fractions.

Follow-up Case Studies

Further study focused on key features of developing mathematical structure, and monitored changes in children's multiplicative reasoning. This is shown through in-depth case study in the fourth year. Twenty-four Year 5 students (aged 10 to 11 years), representing extremes in mathematical ability (twelve high ability and twelve low ability), were drawn from the 3-year longitudinal study. Interview data were combined to form a more coherent picture of these student's key aspects of multiplicative reasoning over a 4-year period. Analysis of interviews was supplemented with work samples of students' representations and classroom assessment records. Interviews were videotaped to ensure the reliability of coding and for in-depth analysis.

Subjects were interviewed four times during the fourth year at regular intervals by a trained research assistant. At the time of final interview twenty one of the twenty-four subjects remained. Tasks initially developed to assess key aspects of multiplicative reasoning were modified for the low ability group of students (Mulligan et al., 1997). Task categories were extended in the fourth year to include fractions and decimals and area concepts. In some cases interview procedures were modified to allow students to further opportunities to demonstrate a variety of responses to the one task. The researcher asked further questions to evoke other available representations from the child.

Analysis

Data were analysed initially for performance across tasks and for individual patterns of strategy use across the four-year period. This paper reports further analysis of the interview data for the twenty-four subjects.

Table 1
Classification of Representations by Imagery, Structure and Nature of Image

CATEGORY	*DESCRIPTION*
Mode of Image	
Perceptual	Objects, figures or gestures used as models by the child which refer to quantities
Pictorial	Pictures drawn or descriptions of objects, events or situations given or described by the child which refer to quantities
Ikonic	Notations such as dots, dashes, tally marks or shapes that represent images of quantity
Notational	Conventional numerals (drawn) or descriptions of numerals represented individually or a series of numerals; signs and symbols e.g., $\times$ $\div$
Type of Structure	
No structure	Objects, pictures or numerals which do not depict equal grouping structure
Partial structure	Objects, pictures or numerals in rows and columns but not a complete pattern of equal groups
Structure	Objects, pictures or numerals in equal grouping patterns, arrays, grids
Nature of the Image	
Static image	The representation is drawn and/or described as a fixed object or notation
Dynamic image	The representation is drawn or described as changing or moving

(Adapted from Thomas, Mulligan & Goldin, in press)

Their pictorial and notational recordings were coded according to three dimensions: (a) the type of representation identified by the perceptual/pictorial, ikonic and notational recordings; (b) the level of structural development and (c) evidence of a static or dynamic nature of the image. Table 1 describes the classifications for each of these dimensions by *mode of representation, type of structure* and *nature of image.*

Discussion of Results

Figures 3 to 6 compare high and low ability students' use of structure and type of representations across tasks and interview stages. Interview Years 1 to 4 refer to grade levels 2 to 5 in New South Wales schools. For the purpose of this summary, one category of 'structure' and one category of 'representation' were reassigned for each student for the first three years and at two interview stages in the fourth year. This was based on the most dominant characteristics shown in their responses and drawn representations across tasks.

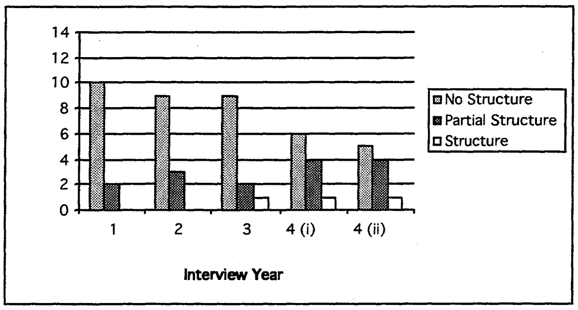

Figure 3. Low ability students' use of structure.

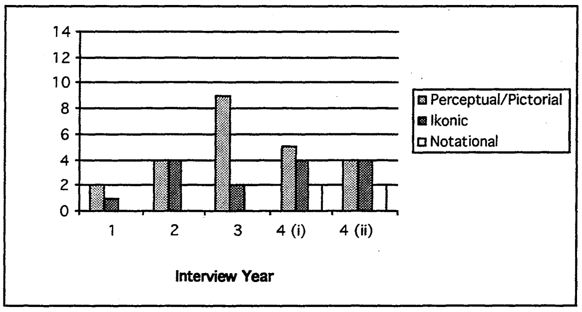

Figure 4. Low ability students' representations.

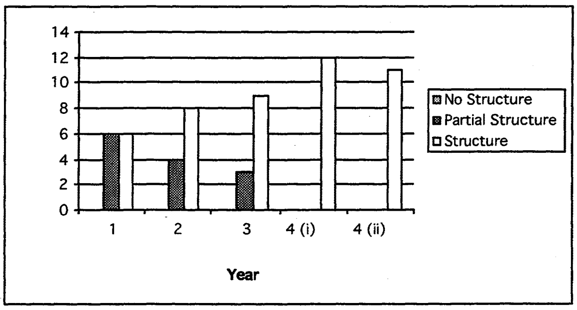

Figure 5. High ability students' use of structure.

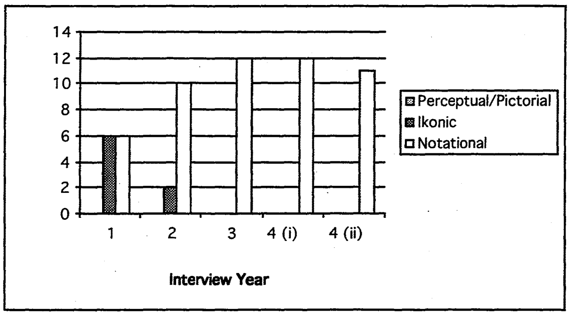

Figure 6. High ability students' representations.

Figures 3 and 4 indicate that the lower ability subjects differed significantly from the higher ability subjects in both their use of structure and their representations. In all but one case their perceptual/pictorial and ikonic representations lacked structural cohesion and mathematical structure depicting equal groups. Another problem detected was an underlying difficulty with basic fractions concepts and appropriate representations of 'part/whole' and 'equal parts'. There was, however, some evidence of 'partial structure' reflected in recordings of equal groups combined with single objects and unitary counting. The absence of underlying structures persisted through to the end of Year 5 for five of these students. Notational representations of multiplicative situations were not forthcoming because these students could not use multiplication and division as operations and could not recall number facts.

Figures 5 and 6 show that the high ability group used notational representations with recognisable mathematical structure. For example, arrays, diagrams, webs or grids were often used in conjunction with number patterns or number sentences, for example 3, 6, 9, 12,... drawn on an array. Most of these representations were reported as having dynamic properties (moving or changing). This was consistent with patterns shown in earlier studies of gifted students (Thomas & Mulligan, 1995). This dynamism showed evidence of underlying mathematical creativity in four of the cases where multiplicative problems were solved using a variety of non-standard, mathematically sophisticated methods.

The use of perceptual/pictorial and ikonic representations for low ability students is not surprising and supports other findings (Mulligan et al., 1997; Thomas et al., in press). The level of representation alone, however, does not explain the inability to initiate and use structure in multiplicative situations. Even when low ability students are given 'models' to promote structure, for example arrays, they are unable to depict spatial organisation or mathematical grouping. In contrast, high ability students indicated existing structures from early Grade 2 (Year 1 of the study), and partial structures for some students progressed to more sophisticated structures by Year 3. This raises the question of why high ability students develop structure even before Grade 2 and whether this reflects high-level visualisation skills or innate mathematical ability.

Implications

Data drawn from twenty-four cases does not permit immediate generalisation. Further longitudinal study needs to focus more explicitly on the origins of both spatial and mathematical structure. It is critical to identify and prioritise those processes that promote the development of structure across a range of mathematical concepts. Part of this identification will involve investigating how young students establish, or fail to establish, relationships between one aspect of mathematical structure and another, such as using a 2 x 10 grid to identify two groups of ten at the same time as recognising the perimeter as 24 units.

The analysis of students' interviews and representations over a four-year period showed some developmental trends that can contribute to a more coherent framework for assessing multiplicative reasoning. It appears that current curriculum contributes to the underlying problems because it lacks emphasis on early development of fractions and multiplicative concepts. Further, can teachers become more aware of these apparent differences and develop strategies for promoting 'structure' across mathematics curricula? In 2002 a new study entitled "Young children's construction and application of mathematical pattern and structure in early numeracy" is examining first graders structural relationships between early number and space concepts while also examining their mathematical representations.

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