The Transition From Concrete to Abstract Decimal Fractions: Taking Stock at the Beginning of 6th Grade in German Schools

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When starting German 6th grade, students already have broad prior experiences with decimal fractions as numbers for measuring quantities. Diagnostic written tests and individual interviews were used to examine how far this experience promotes the transition to abstract decimal fractions shortly before they are taught systematically. Results showed that this transition is still only just beginning in many students, and that the necessary extension of the place-value system is still hampered by incorrect transfers from natural numbers.

German schools teach fractions and decimal fractions systematically in the 6th grade (approximate age 11 years). Before this, students have already gathered initial experiences with simple *fractions* at primary school (Grades 3 and 4) as well as in daily life. At this stage, they have far more experience with decimal fractions in the form of measures of quantity ("concrete decimal fractions"). These are addressed in primary school lessons (Grades 3 and 4) and are also reexamined--to a different extent--in Grade 5. However, they also play an important role in daily life--particularly because, unlike some other countries, the metric system has a long tradition in Germany and is fully integrated into daily life.

How far does this comprehensive experience with concrete decimal fractions lead to the formation of an abstract concept of decimal fractions (i.e., one that does not refer to concrete quantities) by the beginning of 6th grade? Has sufficient basic knowledge accumulated up to this time to provide a rapid and problem-free transition to decimal fractions as abstract arithmetic numbers when they are taught in the coming year? Is this transition even something that occurs almost automatically? Up to now, no empirical findings on these issues are available in German-speaking countries (see, for a related study on fractions, Padberg, 2002). Some interesting studies in Australasia, however, have been stimulating for my work, although they may not address precisely these topics. Particular mention should be given to the work of Baturo and Cooper (2000), Irwin (1999), Moloney and Stacey (1997), Stacey and Steinle (1999), Steinle and Stacey (1998), and Thomas and Mulligan (1999).

The Present Study

My empirical study is based on a specially developed diagnostic test followed by videotaped individual interviews. The diagnostic test (lasting for one 45-min lesson) was given to six complete classes from three different technical secondary schools (in German: *Realschule*) containing a total of 165 students who had just started 6th grade. It addressed the following topics: reading decimal fractions, illustrating decimal fractions, everyday experiences, decimal fractions and scales (lengths, volumes), the decimal place-value system, the local versus global perspective on decimal fractions, ordering decimals, and

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simple decimal arithmetic operations (adding, subtracting, multiplying with natural numbers, measurement and partitive division). For reasons of space, only a small part of this study is reported here.

The videotaped individual interviews were carried out with 13 of these students 2-3 hours after completing the diagnostic test. They took about 10-15 min for each student. Two students were selected from each class (apart from one class in which three students were selected) according to the following criteria: interesting solution approaches in the test, unclear solution strategies, or characteristic error strategies. Because not all parents gave permission for their child to be interviewed, a further selection criterion was parental permission.

The present study is the first, just completed part of a longitudinal research program planned to run for 2 years.

Results

Reading Decimal Fractions

Item 1(item numbers do not correspond to their sequence of presentation in the written text) "How would you read the following number: 3.25? Please circle only *one* answer". Participants could choose between the following answers: "Don't know", "Three hundred and twenty-five", "Three with twenty-five left over", "Three and twenty-five", "Three point twenty-five", "Three point two five", and "Three and one-twenty-fifth". Forty-seven percent gave the answer "Three point two five", and 44% reported "Three point twenty-five". Two percent gave no reply, and 8% gave one of the other replies.

It was reassuring to find that almost one-half of the students circled the correct reading before systematic instruction in decimal fractions, although their only prior experience with them had been in the context of quantities--which would point particularly to the other formulation (e.g., €3.25 = three Euro twenty-five cents). Nonetheless, it is problematic to see that almost all the others simply transferred the way of reading quantities to decimal fractions (a common practice in daily German language). This leads to typical errors such as: (a) 0.4 is smaller than 0.35 because 4 is smaller than 35; (b) 0.27 plus 0.7 = 0.34 because 27 plus 7 = 34; and (c) 0.97 minus 0.6 = 0.91 because 97 minus 6 = 91.

The strong trend toward this problematic reading has to be monitored carefully and countered in the subsequent systematic treatment of decimal fractions, because--as the interviews confirmed--there is a strong degree of uncertainty regarding how to read fractions correctly: one-half of these interviewed students named a different way of reading fractions to that they had given in the written test.

Illustrations of the Decimal Fraction 0.5

Item 2. "Draw a picture illustrating the decimal fraction 0.5 (using a line, a rectangle, a circle, or ...)". This item also presented the option of "I don't understand the task". Results revealed the following strategies:

Strategy 1 (14% of participants). Here, 0.5 was recognizably depicted as one-half of a line or a two-dimensional figure.

Strategy 2 (5%). 0.5 cm was marked out on a longer line with further subdivisions. These students are probably already on the path toward using 0.5 as a marker on a number line.

Strategy 3 (23%). These students drew a 0.5-cm-long line with a ruler (or, partly because of the item description, a square with 0.5-cm sides) and (still) identified 0.5 with 0.5 cm.

Strategy 4 (9%). These students drew only a line, a square, a rectangle, or a circle, sometimes with a length or diameter of 5 cm (avoidant reaction).

Various stages on the path toward a general concept of decimal fractions could be recognized in these four strategies. However, this item indicated that the majority of students still seemed to associate decimal fractions completely with concrete quantities. This is also indicated by the large proportion of no replies (42%). A final 7% percent of students gave other answers.

Decimal Fractions and Scales

To save space, I shall present only two items here with linear scales.

Item 3. "Use an arrow (↑) to mark 4.7 m on the following scale". The item then presented a scale calibrated in tenths from 4 m to 5 m.

This item was answered correctly by 87% of participants. A further 4% marked the scale one place too far to the left or right. Seven percent produced another error, and 2% made no reply.

Item 4. "What measure does the arrow (↑) mark on the following scale". The item then gave participants a scale calibrated in tenths from 3 m to 4 m with an arrow marking 3.4 m.

Ninety-two percent completed this item correctly, 4% gave a wrong answer, and a further 4% gave no reply.

The proportions of correct solutions to these items differed markedly from those for Item 2. Hence, at the beginning of 6th grade, almost all students associated visible representations reporting lengths with simple decimal fractions as measures (although a clear decline in performance was observed when decimal fractions had two places following the decimal point).

Decimal Place-Value Notation

Is the extension of the place-value concept from natural to rational numbers easy for students and thereby almost bound to succeed? The amount of prior work on decimal fractions as concrete quantities would seem to support this assumption. I shall report only two items from the large amount of data gathered in the present study.

Item 5. "How many tenths make up one whole"? Thirty-eight percent answered "10". Nineteen percent gave some other answer, but 43% gave no answer at all. These results speak for themselves (see, also, Baturo & Cooper, 2000). Another item in the test asked how many tens make up one thousand. Only just over 50% of the students managed to give the correct answer to this question on the place-value system for natural numbers--a devastating finding.

Item 6. "Use a cross or a circle to mark the *hundredths* in 7.654". Twelve percent marked the "5"; 28%, the "6"; and 9%, the "4"; whereas 10% gave other answers. A total of 41% gave no answer at all. Hence, despite a higher probability of guessing correctly, only 12% got it right (the proportion of correct solutions in another item asking them to mark the tenths was even lower at 10%).

The answer "6" may be traced back to an incorrect transfer from natural numbers in which the third place from the right (after the decimal point) is viewed as a hundredth--as in natural numbers in which the third place is occupied by the hundreds. This strategy may well also be supported by the fact that the number in this item has exactly three decimal places after the point, and this may encourage participants even more strongly to view the decimals as hundredths, tenths, and, perhaps, "oneths". However, the incorrect way of reading this number as "seven point six hundred and fifty-four" also suggests this error, along with a "point-separates" strategy that I have often observed after systematically teaching decimal fractions (Padberg, 2002). Students applying this strategy view the decimal point as simply separating two natural numbers.

There is a strong probability that the answer "4" is due to another incorrect transfer from natural numbers in which, for reasons of symmetry, the first place after the point is interpreted as a oneth (analogue to the units) followed by tenths and hundredths. The proportion of correct answers is very low on this item and the number of no replies is high, indicating that the extension of the place-value system needs to be dealt with thoroughly at the beginning of the systematic teaching of decimal fractions. The degree of prior knowledge is very low, and it is in no way something that emerges of its own accord (see, also, Baturo & Cooper, 2000). It is also important to contrast decimal fractions with natural numbers, because most of the many incorrect solutions were the outcome of erroneous transfers from natural numbers. This (incorrect) recourse to familiar natural numbers when learning decimal fractions is a frequent problem (see, also, Moloney & Stacey, 1997). However, great care is needed to prevent this misconception from obstinately consolidating (see, also, Moloney & Stacey, 1997; Padberg, 2002; Steinle & Stacey, 1998).

Local Versus Global Perspective on Decimal Fractions

Item 7. "Laura and Alexandra both want to explain the decimal number 0.75 to Sarah. Laura tells her: '0.75 is seventy-five hundredths'. Alexandra says: '0.75 is seven tenths plus five hundredths'. Which explanation is correct? Draw a circle round the name of the girl who has given the right explanation". Possible answers were "Laura", "Alexandra", "Both girls", "Neither girl", or "Don't know". Afterwards, the item asked: "If you have drawn a circle round neither girl, please use a few words to tell us why".

Nineteen percent said that both girls were correct; 22%, Laura; and 28%, Alexandra. Another 19% said that neither explanation was correct, and 12% said that they didn't know. Although answers were distributed fairly equally, the local perspective (Alexandra) seemed to be slightly more dominant than the global one (Laura). Both this equal distribution and the responses in the interviews also indicate that many students gave rather spontaneous decisions that they were unable to justify. The justifications for the answer "Neither" are particularly interesting, although only one-half of these students completed this part of their item. The most frequent argument was that 0.75 is composed of seven tenths and five units (oneths). This, like the answer, "0.75 is 75 *tenths*", reveals two different incorrect transfers from the domain of natural numbers (see the previous section on place-value notation). The answer 0.75 = 75 cm confirms that these students (still) identified 0.75 with 0.75 m.

Decimal Fractions "Between" Decimal Fractions

From the broad range of questions on ordering decimals, I shall report just one test item.

Item 8. "Can you name a decimal number that is larger than 1.5 and smaller than 1.6"? (This open question also provided the response option "No".)

Correct answers were given by 37% of the students; 6% produced other answers (see below); 16% produced incorrect answers; and 42% replied "No". The most frequent correct solution was the intermediate number: 1.55. The "Other" category contained rather creative replies that did not use standard notation (the most frequent was 1.5 1/2 followed by 1.5.5 and 1.5.1). Some of the justifications for "No" in the interviews were that there is no decimal number between the two, or that, although such numbers exist, the student did not know how to name them. There was a wide range of incorrect answers.

Some additional remarks. (a) A comparable item using *lengths* ("Mary wants to buy a new book shelf. It needs to be longer than 2.4 m and shorter than 2.5 m. Give an example of how long the book shelf should be".) led--contrary to my expectations--to only a slight increase in the proportion of correct solutions (from 37% to 42%), an equal proportion of nonstandard replies, and a markedly lower number of students who were unable to report an in-between number (from 42% to 32%), nonetheless, at the expense of a slight increase in the proportion of incorrect answers (from 16% to 20%). (b) Compared with a corresponding item using simple fractions (Padberg, 2002), far more students at the beginning of 6th grade associated more or less intuitive ideas with decimal fractions. In the study of fractions, only 5% of a comparable sample were able to report a correct fraction between one quarter and one half. Alongside the greater amount of prior preparation, this may also be due, in part, to the decimal notation being more familiar than that of (simple) fractions.

Discussion

Regarding their knowledge of decimal fractions, students at the beginning of 6th grade are going through a transitional phase between concrete decimal fractions as measures of quantity, which they have known for quite a while in both daily life and school, and the abstract decimal fractions as abstract computational numbers, which will be taught systematically during the course of this school year.

The study provides the following picture of their progress along this path: A high percentage of students are able to read decimal fractions (see Item 1), although almost one half of them use a common form of reading in daily life that leads to a variety of characteristic errors. They also have a confident command--at least in simple cases--of the notation of decimal fractions as measurements of quantity in scales (see Items 3 and 4). However, things are completely different when students are asked about the pictures they associate with simple decimal fractions such as 0.5. Although perhaps 20% of the students in this sample are already approaching the abstract concept of the decimal fraction, many still identify decimal fractions with specific quantities (in this case, frequently 0.5 cm), whereas many others are uncertain and give no answer (see Item 2). A good indicator of progress on the path toward an abstract concept of the decimal fraction is also the item on decimal fractions between two given decimal fractions, an item that is probably easier than illustrating decimal fractions (see Item 8). Almost 40% can report at least one decimal fraction here (a percentage that hardly increases in a parallel item with quantities). In contrast, the students still have hardly any experience with the two *perspectives* on decimal fractions (local vs. global, see Item 7). Their decision is reached more impulsively rather than on the basis of any plausible justification. The knowledge on extending the decimal place-value system (see Items 5 and 6), which is so necessary for a comprehension of decimal fractions, is still extremely overlaid by--what are here incorrectly transferred--prior experiences on the place-value notation of natural numbers. Hence, on two items, only about 10% of students mark the tenths or hundredths correctly. A very dominant misconception here is that the point in decimal fractions separates two natural numbers. As a result, it is not the point but the last place to the right that forms the reference basis in decimals as well (moreover, this misconception is supported additionally by the problematic way of reading decimal fractions mentioned above in Item 1). Furthermore, one can frequently observe the idea that decimal fractions following the point are constructed symmetrically to the whole numbers before the point (oneths). A high degree of uncertainty is also documented once again by the fact that approximately 40% of students are unable to solve this item. Fractions--at least in the form of fractions with powers of 10 as denominators--form a further important component of the concept of the decimal fraction that I am unable to discuss here because of lack of space (see Padberg, 2002, in press).

The study confirms very clearly that a confident mastery of concrete decimal fractions should not be equated in any way with a good knowledge of abstract decimal fractions. It is far more the case that the corresponding transfer needs to be carried out carefully and gradually--and not just quickly in one lesson. Great care is necessary when extending the place-value system. For example, the relations between the place values need to be discussed not just in the one direction but in both. The relations between non-neighboring place values and between place values to the left and the right of the decimal point need to be considered, to name just a few perspectives (see, also, Baturo & Cooper, 2000). The commonalities with natural numbers should not be overemphasized; it is specifically the *differences* that have to be worked out clearly. If this is not done with great care, it is easy for students to acquire misconceptions regarding decimal fractions that may well persist for a long time (see Padberg, 2002; Stacey & Steinle, 1999; Steinle & Stacey, 1998). However, a good and careful foundation in decimal fractions has a very high priority in our society with its widespread use of pocket calculators and computers as well as its daily use of the metric measurement system. When summarising the outcome of their wide-ranging own studies, Wearne and Hiebert (1988, p. 223) rightfully confirm that

> Many difficulties can be traced to an incomplete or nonexistent understanding of the meaning of the written symbol ... Without semantic meanings for the symbol, students have little choice but to memorize syntactic rules that prescribe how to manipulate symbols.

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